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All parabolas similar? Never!

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Introduction

Some time ago during the 1989 Annual congress of MASA in Pretoria the author completely stunned the audience with the announcement that *'all parabolas are similar'*. Many also did not seem to believe the intuitive explanation that was given. That this result is probably unknown to most people who have passed through the South African Senior Secondary Mathematics Curriculum is borne out by the following initial remark by a senior mathematics professor upon being told about it by me:

> *"All parabolas similar? Never! ... But of course it depends on what your definition of similarity is ..."*

However, this result is true without having to divert from the *'usual'* definition of similarity and using some sort of conjuring trick and *'weird'* definition!

While preparing the abovementioned talk on Transformation Geometry and considering its applications to other parts of the mathematics curriculum, one of the questions the author considered worthwhile discussing was the study of the effect of various transformations on the graphs presently prescribed in the syllabus (see references). This would then naturally lead to the question of what happens to the equation of a graph in general under, for instance, the basic **isometric** transformations, namely, reflections, translations and rotations.

By first looking at specific cases, children may, for example, discover (and explain) for themselves that the equation $y = f(x)$ under a reflection in the Y-axis changes to $y = f(-x)$ and a translation of a units in the x-direction changes it to $y = f(x - a)$. In contrast to the isometric transformations which preserve congruence, we have other transformations which could be studied at school like the **similarities** (technically, the dilations) which preserve similarity (shape) and the affinities which preserve the parallelness of corresponding lines. Basically the similarities consist of **enlargements** or **reductions**, while familiar affinities used at school, especially in trigonometry, are **stretches** either in the x or y direction. For example, $y = 2\sin x$ represents a stretch of $y = \sin x$ with a factor 2 in the y-direction, while $y = (2x)^3 - 4(2x)$ represents a stretch (shrink) of $y = x^3 - 4x$ in the x-direction by a factor ½.

Definitions for similarities and stretches

Let's first briefly concentrate on what a similarity is. Consider the square with the point $P(x;y)$ as shown in Figure 1 being mapped with an enlargement from the origin on to the larger square with $Q(x';y')$ the image of $P(x;y)$. Since triangles OPR and OQS are similar it follows that $x'/x = k$ and $y'/y = k$, and this relationship is true between all points of the smaller square and the larger square. Generally therefore, we define an enlargement (similarity) from the origin as the mapping of a point $P(x;y)$ into the point $Q(x';y')$ under the transformation:

$$x' = kx$$
$$y' = ky$$

Note that in the case where $0 \leq |k| < 1$ a *'reduction'* occurs. (By the way, what happens if k is negative?). As can be seen from the example in Figure 1, the similarities are those transformations that preserve shape, since corresponding angles are congruent and the ratios between corresponding sides remain constant. Also note that the isometric transformations can be viewed as those similarities with $|k| = 1$. Of course enlargements need not occur only from the origin as centre, but can occur from any point in the plane. It is very easy to verify

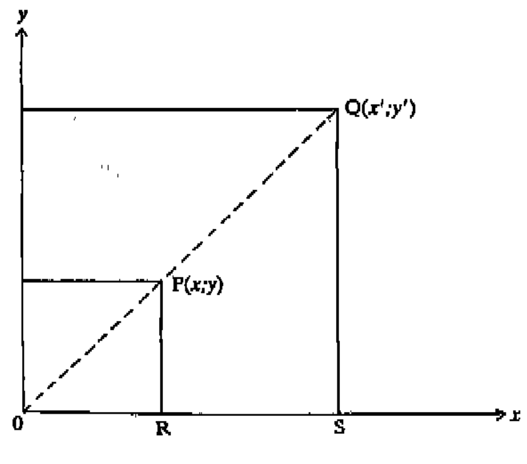

Figure 1

in the same way as in Figure 1 that an enlargement from a point $(p;q)$ is given by

$x' = k(x-p) + p$
$y' = k(y-q) + q$

(In fact one can merely view it as a translation of the centre of enlargement from the origin to the point $(p;q)$). In contrast to an enlargement, a stretch transformation in the y-direction from the origin as centre as shown in Figure 2, is defined by the transformation:

$x' = x$
$y' = ky$

and similarly for a stretch in the x-direction (from the origin as centre):

$x' = kx$
$y' = y$

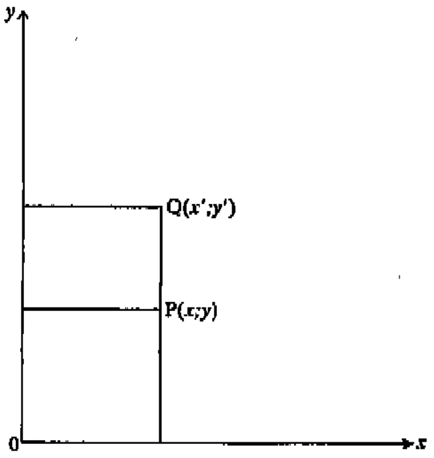

Figure 2

Note that an enlargement can be viewed as two simultaneous stretches with the same value of k, one in the x-direction and one in the y-direction. As also shown by this example stretches in general do not preserve similarity.

Now if we want to see what happens to the equation of a graph $y = f(x)$ under an enlargement from the origin, we simply solve x and y from the corresponding transformation equations and substitute these into $y = f(x)$ to obtain the transformed equation:

$y' = k\,f(x'/k)$ or simply $y = k\,f(x/k)$

For example, $y = 2\sin(x/2)$ implies an enlargement of $y = \sin x$ by a factor of 2. Similar to the above we have:

$y = k\,f(x)$ for a stretch in the y-direction
$y = f(x/k)$ for a stretch in the x-direction.

Some examples

While preparing my MASA talk, some of the examples of transformations on parabolas I considered using, were the stretch of $y = x(x-4)$ by a factor 2 from the origin in the y direction as shown in the first figure in Figure 3 and then its enlargement also by a factor of 2 from the origin as shown by the second figure in Figure 3. One can easily see that the parabola $y = x(x-8)/2$ is similar to $y = x(x-4)$, since its roots and turning point are twice that of the original. But what about $y = 2x(x-4)$? Is it similar to $y = x(x-4)$ and $y = x(x-8)/2$, i.e. does it have the same shape? Certainly not, one would think just looking at its roots which are the same as those for $y = x(x-4)$ while its turning point in contrast, is twice that of the original. After all, stretches in general do not preserve similarity as pointed out earlier.

Although I recalled in the back of my mind at this time some fleeting references in British school textbooks (eg. SMP) that all parabolas were similar, I dismissed these statements (which were given without further ex-

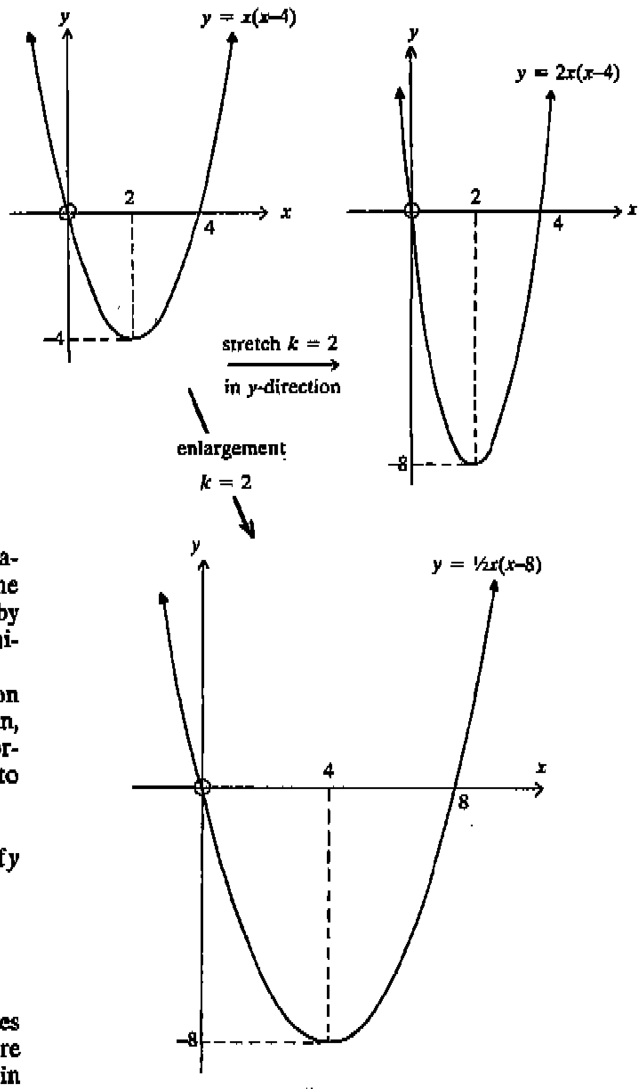

Figure 3

planation or proof) as a sort of general non-mathematical statement like *'all triangles are similar'*. In this sense, the word similar is not used in the strict mathematical sense or to refer to the individual shapes of triangles but merely to indicate that all triangles are *'similar'* in the sense that they all have 3 sides and 3 angles. Similarly, I interpreted this statement merely to mean that generally, parabolas are *'similar'* in the sense that they are all reflective symmetrical around some vertical axis or other and that they all have maximum or minimum points.

A startling discovery

Somewhat later while I was playing around and experimenting with various transformations on graphs, I considered the enlargement from the origin of $y = x^2$ by a factor 2 as follows:

$y = 2(x/2)^2 = ½x^2$

Good grief, it hit me! An enlargement of factor 2 for a parabola is equivalent to a stretch in the y-direction with a factor 1/2. Or similarly, an enlargement of factor 1/2 for a parabola is equivalent to a stretch in the y-direction with a factor 2. Therefore, in the case of parabolas, stretches in the y-direction seem equivalent to certain corresponding enlargements. But does this really make sense? What does it mean?

It took me only a short while to chew this over to realize that this meant that all parabolas are similar, ie. they have exactly the same shape, the only thing that differs is their enlargement. Consider, for instance, the family of parabolas shown in Figure 4 where $y = x^2/2$ is the enlargement of $y = x^2$ by a factor of 2, which in turn is the enlargement of $y = 2x^2$ also by a factor of 2. The similarity between the above parabolas and the enlargements involved become even clearer when we place, for instance, $y = 2x^2$ and $y = x^2$ as shown in Figure 5. Here we can clearly see the parallelness of the respective arms of the two parabolas. Also shown is a rectangle zooming in on part of $y = 2x^2$ which when enlarged by a factor of 2 gives us the enlarged rectangle enclosing the part of $y = x^2$ as shown.

Proof of parabola similarity

Perhaps we should also briefly look at what we exactly mean by the concepts of *'congruence'* and *'similarity'*, especially in terms of transformation geometry. We can say that two figures are congruent, when we can exactly map the one onto the other using any combination of only the isometric transformations (e.g. reflections, rotations and translations). Similarly, two figures are similar when the one can be mapped exactly on to the other using any combinations of the similarity transformations (eg. enlargements, reductions and isometric transformations).

It is now easy to see that all parabolas are similar, since any parabola of the form $y = ax^2 + bx + c = a(x + b/2a)^2 - \Delta/4a$ can be translated so that it lies symmetrically with respect to the Y-axis and with its turning point at the origin, simply by moving it $-b/2a$ units in the x-direction and $\Delta/4a$ in the y-direction. Thus any parabola can be transformed to $y = ax^2$ which upon an

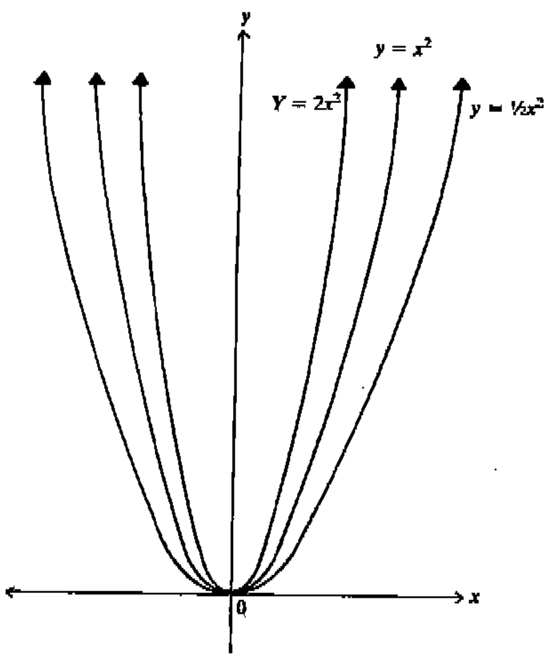

Figure 4

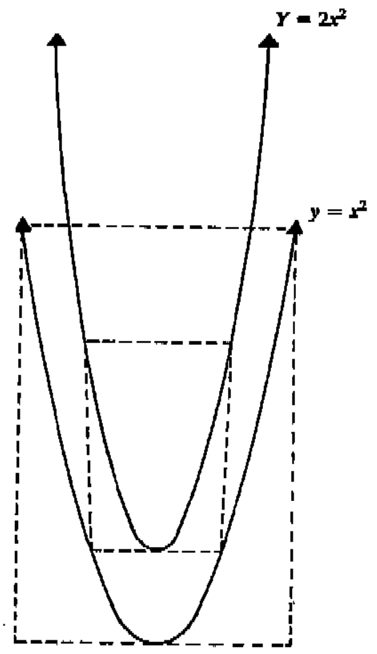

Figure 5

enlargement of $k = a$ from the origin, gives us $y = a \times a(x/a)^2 = x^2$. This means that any parabola of the form $y = ax^2$ is similar to $y = x^2$, and completes the proof. Note that a stretch of $k = a$ of a parabola in the y-direction is therefore equivalent to an enlargement of $k = 1/a$ (and vice versa). It should perhaps also be mentioned that a stretch of $k = 1/\sqrt{a}$ of a parabola in the x-direction is equivalent to a stretch in the y-direction of $k = a$.

According to the above proof the parabolas $y = 2x(x - 4)$ and $y = x(x - 4)$ considered earlier are therefore similar. Instead of translating both parabolas to the origin before enlarging, we could simply translate $y = 2x(x - 4)$ vertically by 4 units so that its turning point coincides with that of $y = x(x-4)$ and then enlarge it with $k = 2$ from the point (2; -4). This is, however, left as an exercise to the reader and/or an addict of algebraic manipulation. (The formula for an enlargement of $y = f(x)$ from a point $(p;q)$ is $y = kf([x + p(k-1)]/k) + q(1-k)$).

In general, South African textbooks have in the past not at all treated the matter of the similarity of the parabolas. In fact, their treatment has usually been mathematically incorrect. For instance, incorrect statements like *'the shape of a parabola $y = ax^2$ depends on the value of a ...'* are commonly used. The correct way would be to say *'the order of magnification (enlargement) of a parabola $y = ax^2$ depends on the value of a'*. Changing the value of a therefore does not change its shape, only its order of magnification.

The similarity of curvilinear figures

After convincing the abovementioned professor about the similarity of the parabolas by giving him the above proof, he remarked:

> *"Yes, I now agree with you that they are similar from a mathematical viewpoint, but certainly not from an everyday viewpoint. Anyone who makes ashtrays or cuts lenses would tell you that the parabolas are not similar."*

I spent quite some time reflecting on this observation before realizing that it probably subconsciously refers to the differences in the curvatures of the parabolas. For example, the curvature of $y = x^2$ is less than the curvature of $y = 2x^2$. (Note that a measure of curvature is given by the derivative, ie. how quickly the gradient changes over Δx. Technically, it is the rate of turning of the tangent in relation to the arc length of the curve). Of course, such differences in curvature are of great importance when one is cutting lenses, making ashtrays or reflectors. However, differences in curvature between figures and curves do not necessarily imply that they are dissimilar. The simplest example is the family of circles, all clearly similar, but the larger the circle, the smaller its curvature.

Talking to Piet Human and Hanlie Murray about the similarity of the parabola, we eventually concluded that the main reason for its *'counter-intuitiveness'*, was probably to be found in the fact that our basic conception of similarity is usually limited to only rectilinear figures; eg. triangles, quadrilaterals, etc. A specific cognitive restructuring and/or accommodation therefore needs to take place before the similarity of curvilinear figures such as the parabola will start making sense.

Finally, another conception which might possibly hamper the extension of the concept of similarity to curvilinear figures is that in the case of similar rectilinear figures there are always corresponding linesegments of the similar figures which are congruent. For example as shown in Figure 6, sides AB and BC of the rectangle ABCD can be placed on top of sections of the sides EF and FC of the larger similar rectangle EFCH. (In fact, this can also be done for non-similar rectangles — so it is a necessary but not a sufficient condition for similarity of rectilinear figures). However, in the case of similar curvilinear figures such as circles or parabolas as also shown in Figure 6, this is not possible since no segments of their curves are congruent.

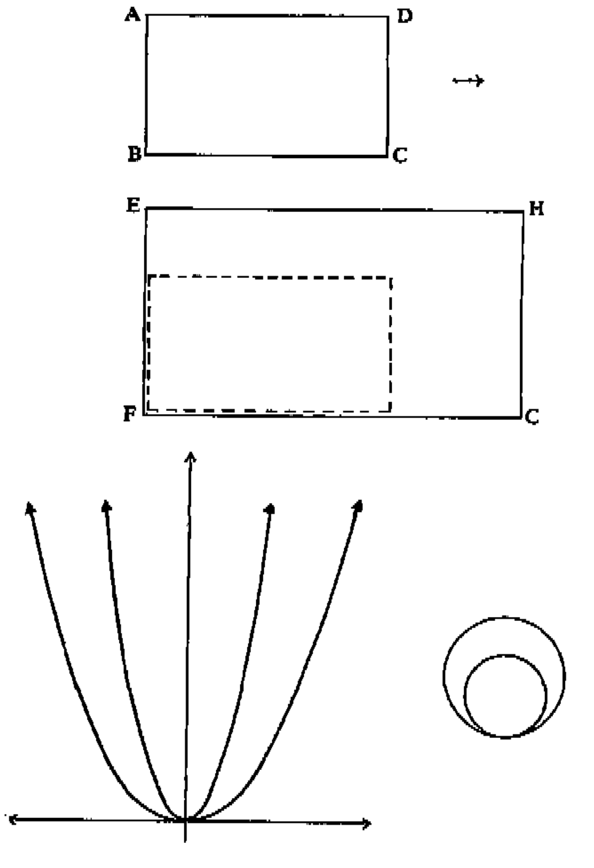

Figure 6

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"Let us begin with Archimedes. We all know the story ('Eureka! Eureka') of his leap from the bath as the law of displacement was demonstrated before his delighted eyes; those who know a little more about him can say how he died.

He was intent on tracing a theorem in the sand, so intent that he did not see the Roman soldier stooping over him, nor hear the soldier's barked command to identify himself, whereupon the soldier killed him. (We can pause here to mourn the further discoveries he might have made, but only if we keep in mind one of the post powerful truths of history, demonstrated perfectly in this case, that death is more powerful than murder: the name of Archimedes will be known for ever, but the name of his killer was forgotten even before the sage was buried.)

It was a good way to die, was it not? For his death came from the power of his concentration on his geometry, and that concentration came in turn from his belief that the truths of mathematics, which are eternal, are more important than war and peace, which are ephemeral."

— Bernard Levin, in the **Times**, 28 January 1991.