An extension of the six-point circle theorem for a generalised Van Aubel configuration

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Introduction

We deal with an extension of the six-point circle theorem for the quadrilateral [1] when the Van Aubel configuration is generalised as in [2] and [3]: similar parallelograms are constructed, all internally or all externally, on the sides of a given quadrilateral.

Van Aubel's theorem and the six-point circle

We briefly recall Van Aubel's theorem and the six-point circle theorem.

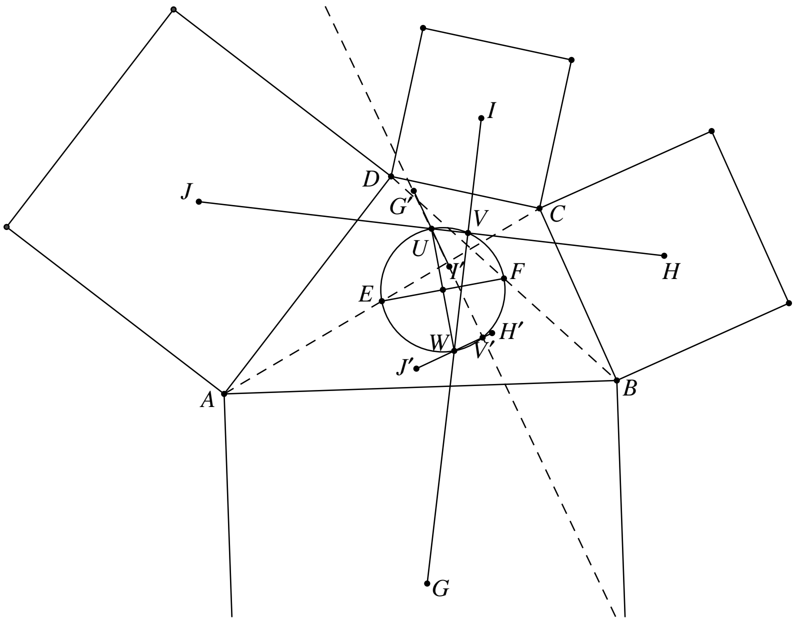

FIGURE 1: Van Aubel's theorem and six-point circle

Theorem (Van Aubel's theorem [4])

Given a convex quadrilateral, on each side construct a square externally to the quadrilateral. Join the centres of the squares constructed on the opposite sides. Then the segments obtained are of equal length and orthogonal to each other.

The theorem holds true also for re-entrant quadrilaterals [5], or when the squares are constructed internally to the given quadrilateral.

For crossed quadrilaterals, the theorem holds true when the constructions are carried out in a more general way as shown in [1]: care should be taken in constructing the squares, as *external* and *internal* constructions are not definable in this case. Looking at Figure 1, for the external construction, the equal and orthogonal segments are represented by GI and HJ, and they intersect at V (the first or outer Van Aubel point [1, 6]). For the internal construction, the equal and orthogonal segments are represented by $G'I'$ and $H'J'$, and they intersect at V' (the second or inner Van Aubel point) on the extension of segment $G'I'$.

The Van Aubel segments GI and $H'J'$ bisect each other at W, and HJ and $G'I'$ bisect each other at U, as proved in [1].

In [1], Pellegrinetti proved the following additional property:

Theorem: The midpoints E and F of the quadrilateral's diagonals, the first and second Van Aubel points V and V', and the midpoints U and W of the Van Aubel segments all lie on a circle. Segments EF and UW are two mutually orthogonal diameters of the circle.

A trivial consequence is that the centre of the circle coincides with the quadrilateral's vertex centroid [7]. Just as for Van Aubel's theorem, this theorem holds true independently of the quadrilateral type.

The nomenclature used in [1], although helpful, is not fully correct: formally, a quadrangle does not have diagonal lines. For example, the Gauss-Newton line, defined as the line through the midpoints of the quadrilateral's diagonals, is not definable for a quadrangle.

An extension of Van Aubel's theorem – a synthetic proof

Van Aubel's theorem has been extended to similar rhombi and similar rectangles on the sides in [8], and an extension to similar parallelograms was mentioned, without proof, in [2]. In [8], synthetic proofs are presented while Silvester develops a more general algebraic proof in [3] for similar parallelograms on the sides. Let $\angle(PQ, RS)$ represent the angle between two segments PQ and RS, that is $\angle(PQ, RS) = \angle QPT$ where $\vec{PT} = \vec{RS}$. With this definition $\angle(SR, PQ) = \pi - \angle(PQ, RS)$.

Theorem

If directly similar parallelograms $AMNB$, $CBOP$, $RDCQ$ and $STAD$, with respective centres G, H, I and J are erected on the sides of a given quadrilateral $ABCD$, then $JH/IG = AB/AM$, and $\angle(IG, JH) = \angle MAB$ (see Figure 2, where we have drawn $ABCD$ convex, and the parallelograms are erected externally).

Proof:

Let $AB/BN = \lambda$, $\angle MAB = \alpha$ and $\angle ABN = \beta = \pi - \alpha$. Then, because $AMNB$ and $CBOP$ are similar, we have $OB/BC = \lambda$ and $\angle OBC = \beta$. Thus there is a spiral similarity s with scale factor λ^{-1} and angle β such that $s(B) = B$, $s(A) = N$ and $s(O) = C$. It follows that $AO/NC = \lambda$ and $\angle(CN, AO) = \alpha$. Let E be the midpoint of AC. Then, since G, E, H are the midpoints of AN, AC, CO, respectively, if we apply the triangle midsegment theorem to ΔACN and ΔACO, we deduce that $\angle GEH = \alpha$ and $EH/EG = \lambda$.

By a similar argument, since I, J are the midpoints of CR, AS, respectively, then $EJ/EI = \lambda$ and $\angle IEJ = \alpha$. So there is a spiral similarity s' with scale factor λ and angle α such that $s'(E) = E$, $s'(G) = H$ and $s'(I) = J$. It follows that $JH/IG = \lambda$ and $\angle(IG, JH) = \alpha = \angle MAB$, as required.

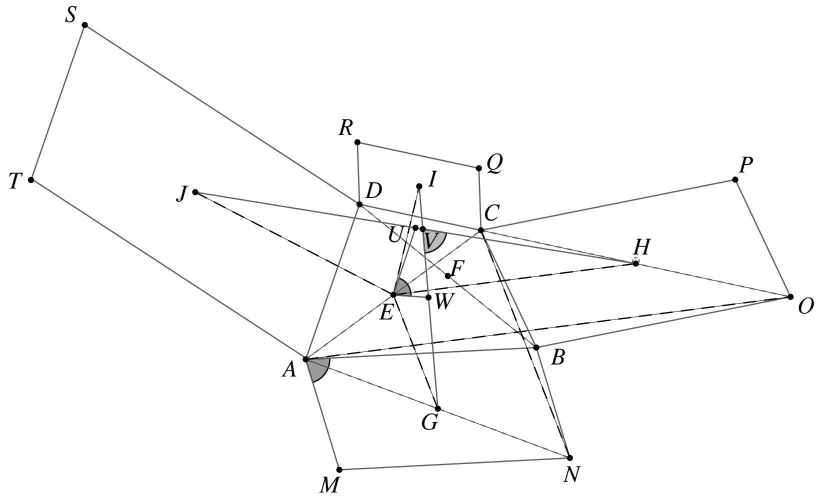

FIGURE 2: Extension of Van Aubel's theorem

Just as for Van Aubel's theorem, this theorem is quite general. It holds true when the parallelograms are erected internally on the sides of the given quadrilateral and no matter if this latest is convex, re-entrant or even crossed (in this last case, external/internal ceases to have any meaning − see previous section). The configuration for the internal parallelograms is represented in Figure 3. The internal parallelogram centres are produced reflecting the centres of the externally erected parallelograms with respect to the sides of the given quadrilateral. So, in such a configuration, the internal parallelograms (not shown in the Figure) are symmetric to the external ones with respect to the quadrilateral sides, so that the angle formed by the sides of these internal parallelograms equals α. We invite readers to draw other possible configurations where two of the points A, B, C, D coincide, or three of them or even all four are collinear, for example.

Extending the nomenclature provided in [1], point V (in Figure 2) and point V' (in Figure 3) will be referred to as generalised Van Aubel points. The segments JH, IG, $J'H'$ and $I'G'$ will be referred to as generalised Van Aubel segments.

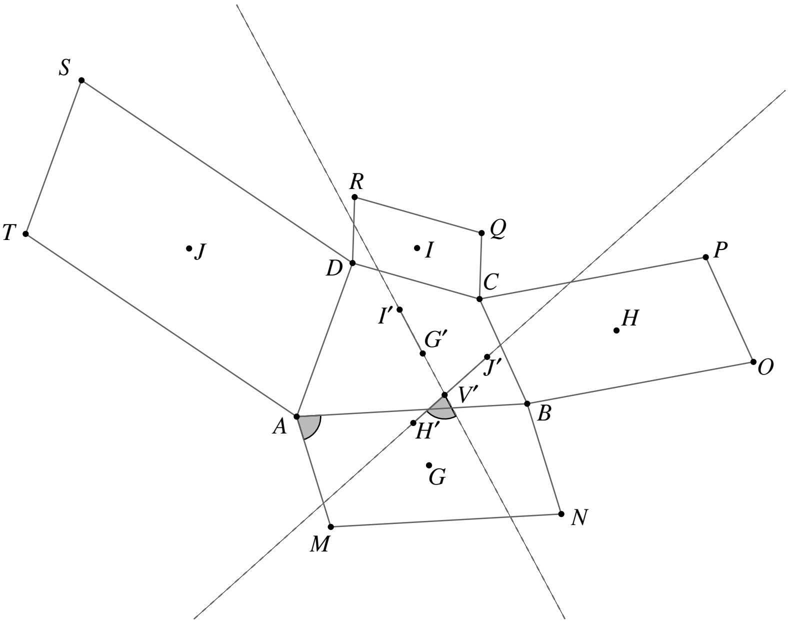

FIGURE 3: Extension of Van Aubel's theorem, internal construction*

A new class of five-point circles

Looking again at Figure 2, let the midpoints of the generalised Van Aubel segments, JH and IG, be U and W, respectively, and let F be the midpoint of the quadrilateral's diagonal BD. We state:

Theorem

Points E, F, W, U and V are concyclic.

Proof

Since the spiral similarity s' (as above) sends ΔGEI to ΔHEJ, it also sends the median EW to the median EU, and therefore $\angle WEU = \alpha$. E and V are on the different sides of WU. Applying the converse of proposition 22 from the third book of Euclid's *Elements* [9, 10] to quadrilateral $EWVU$ (a convex quadrilateral is cyclic if, and only if, its opposite angles are supplementary – or, equivalently, if, and only if, an exterior angle is equal to an opposite internal angle), points E, W, V and U are concyclic. For the

* For clarity, we have drawn the external parallelograms rather than the internal ones. The centres of the internal parallelograms G', H', I' and J' are the reflections of the centres of the external parallelograms in the respective sides.

proof to hold for any possible configuration (different values of α and/or λ and/or a different quadrilateral *ABCD*), we must handle the case in which E and V are on the same side of WU. In this case, WU is subtended by the equal angles $\angle WEU$ and $\angle WVU$. So applying the converse of proposition 21 from the third book of Euclid's *Elements* [9, 10] we deduce again that E, W, U and V are concyclic. Similarly, points F, W, U and V are concyclic, whence the result.

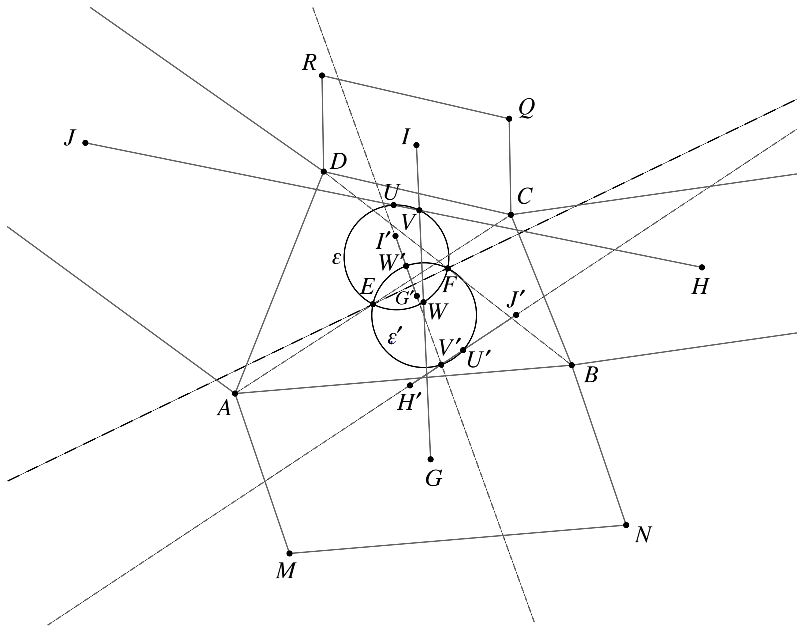

FIGURE 4: Two symmetric five-point circles for the symmetric external and internal generalised Van Aubel configurations

This five-point circle, ε, is represented in Figures 4 and 5.

For different values of α and λ different five-point circles can be obtained, but they all pass through E and F. This way, a system of coaxial circles with the Gauss-Newton line as common chord is generated. Again, this theorem is fully general: in Figure 4 we also show the five-point circle ε' corresponding to the internal configuration with the internal parallelograms symmetric to the external ones with respect to the quadrilateral sides. We notice that circles ε and ε' seem symmetric with respect to the Gauss-Newton line of the quadrilateral. We also wonder if different configurations produce again the ε circle. The following theorem resolves these latest points and opens the door to a few additional results.

Theorem

There is a spiral similarity k between ΔADJ and ΔEFU such that $k(A) = E, k(D) = F, k(J) = U$.

Let γ be the circle through A, D, J; we have $k(\gamma) = \varepsilon$. Similarly, there is a spiral similarity, k', between ΔCBH and ΔEFU such that $k'(\gamma') = \varepsilon$ with γ' the circle through B, H, C (see Figure 5). For the proof we will exploit the following lemma presented and proved in [11]:

Lemma: If the corresponding vertices of two directly similar figures are connected, then the midpoints of those connecting segments form another figure, similar to the other two.

Proof: Because the parallelograms constructed on the sides AD and BC are similar by hypothesis, there is a spiral similarity h between ΔADJ and ΔCBH such that $h(A) = C, h(D) = B$ and $h(J) = H$. Because E, F, U are the midpoints of AC, DB, JH, respectively, applying the lemma we deduce that there is a spiral similarity k between ΔADJ and ΔEFU, such that $k(A) = E, k(D) = F, k(J) = U$.

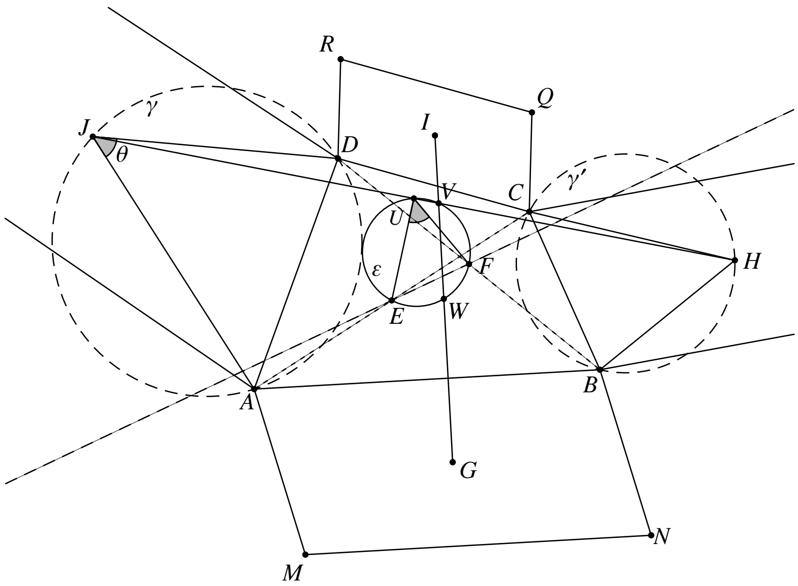

FIGURE 5: A revealing proof

This proof enables the computation of the radius of circle ε. For by the similarities, $\angle EUF = \angle AJD = \angle CHB = \theta$, say, which is the angle between the diagonals of the parallelogram. Then EF subtends an angle 2θ at the centre of ε, whence its radius is $\dfrac{EF}{2\sin\theta}$.

Also, it follows that, if the parallelograms are altered by moving J to some other point of the same circle γ, then ε is invariant. And it also follows that, if the parallelograms are reflected in the various sides of the quadrilateral, then γ is reflected in AD and hence ε is reflected in the image of AD, that is in EF (see Figure 4). So the symmetry relation between ε and ε' around the Gauss-Newton line is explained. We notice that circle ε is equal to its reflection if, and only if, EF is a diameter, that is if, and only if, AD is a diameter of γ. This means that $\theta = \pi/2$ or $AJ \perp JD$, which happens if, and only if, the parallelograms are rhombi. In other words, in such a case, circle ε becomes the six-point circle for the quadrilateral. It follows that the six-point circle can be traced as the locus of the the generalised Van Aubel points or the midpoints of the generalised Van Aubel segments when the rhombi are continuously altered. The dynamic geometry sketches at [12] illustrate these last results.

A few concluding remarks

Interestingly, for the internal configuration in which the sides ratio of the parallelogram is inverted with respect to the sides ratio of the external configuration and the angles between the parallelogram sides are kept as for the external configuration ($AB/AM' = \lambda^{-1} = (AB/AM)^{-1}$ and $\angle BAM' = \alpha$ or else $\pi - \alpha$), we get the same circle as for the external configuration because point J' belongs to the same circle γ. Finally, whenever A, D, S and T are collinear, circle ε degenerates onto the Newton-Gauss line of the $ABCD$ quadrilateral: k maps $\gamma = AD$ onto ε and $\theta = \pi$ or 0 (depending whether J is between A and D or not), so that $\sin\theta = 0$ and the radius of ε is infinite.

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