From De Villiers, M. (1994/2009). Some Adventures in Euclidean Geometry.
Morrisville, NC: Lulu Publishers. pp. 76-88.

Chapter 6

Generalizing Varignon's theorem

One of the most fascinating aspects of mathematics, as already pointed out in Chapter 3, is that it is an evolutionary process in the sense that solutions to problems often lead to further problems, provided one retains an enquiring and open mind to continue asking questions. As Daltry (1966:20) put it: *"..the end of problem-solving may not be solutions so much as new problems."* A particularly useful problem posing strategy is to always consider the possibility of generalizing a particular result, no matter how trivial it may seem, or even when a conjectured generalization turns out to be false.

What follows is an illustrative example of the application of this strategy to Varignon's theorem as illustrated in Figure 81, namely, that if the midpoints E, F, G and H of the adjacent sides of any quadrilateral ABCD are consecutively connected, then EFGH is a parallelogram (also see *Questions and Problems 3,* no. 5). According to Coxeter & Greitzer (1967:53), the first known published proof of this rather simple result was only given in 1731 by Pierre Varignon, and the inscribed parallelogram is consequently often referred to as the Varignon parallelogram. It should again be pointed out that the investigations which follow in this chapter are particularly suited for exploration or demonstration on dynamic software like *Cabri Géomètre* or *Geometer's Sketchpad*.

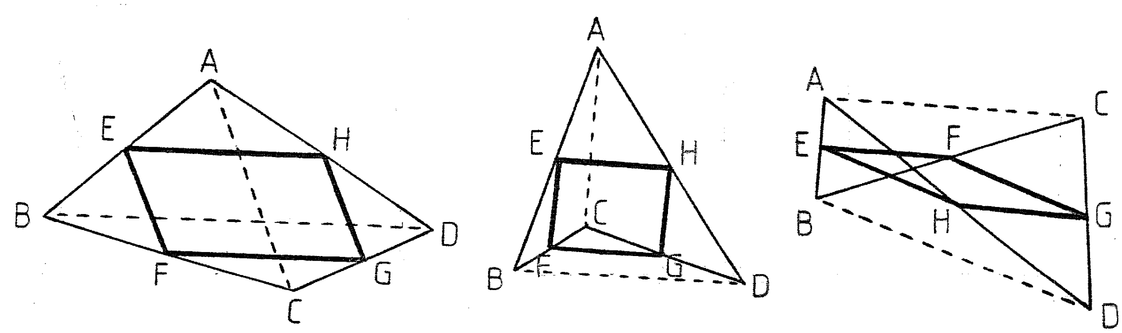

Figure 81

How can we generalize this result? What about generalizing it to other polygons? Is it really necessary that E, F, G and H are midpoints of the sides? Or phrased differently, how can we maintain the result if E, F, G and H are not midpoints?

A first generalization

Let us first look at the generalization to other polygons. Since the result deals with a quadrilateral which has an even number of vertices, we skip the pentagons and firstly consider the possibility of an analogous result for hexagons. If we consecutively connect the midpoints of the adjacent sides of a regular hexagon, we find another regular hexagon as shown in Figure 82a. In other words, we obtain a hexagon with opposite sides parallel and equal, i.e. a *parallelo-hexagon*. (Also see Solutions 2, no. 17). Of course, this is not true in general, for if we repeat the construction with the hexagon shown in Figure 82b, we do not find a hexagon with opposite sides parallel and equal. The question now arises: under which conditions would we find a hexagon with opposite sides parallel and equal?

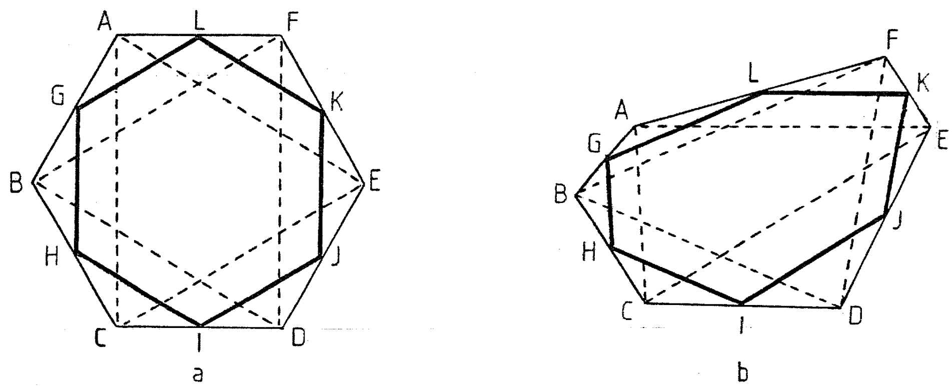

Figure 82

Well, let's just for a moment reflect back on the deductive explanation for the quadrilaterals and the characteristic property upon which it depends (Figure 81). By drawing a diagonal (or both), we simply utilize the theorem that the line segment connecting the midpoints of two sides of a triangle is parallel and equal to half the length of the third side. Now bearing this explanation in mind, and looking at the hexagon in Figure 82b, it should immediately be clear that GH would be parallel and equal to the opposite side JK if AC //= DF. Similarly, it follows that BD //= EA ⇒ HI //= KL and CE //= FB ⇒ IJ //= LG. In other words, in any hexagon with the aforementioned properties we would find an inscribed parallelo-hexagon.

What is more, a hexagon ABCDEF with AC //= DF, BD //= EA and CE //= FB, would itself be a parallelo-hexagon. For example, AC //= DF ⇒ ACDF is a parm ⇒ AF //= CD. The same is true for the other two pairs of opposite sides. Three examples of parallelo-hexagons with the midpoints of their sides consecutively connected to form inscribed parallelo-hexagons are shown in Figure 83. Note that the parallelo-hexagons ABCDEF can be drawn by constructing two congruent triangles ACE and DFB with AC//DF, AE//DB and CE//FB, and then connecting A, B, C, D, E and F. (Also compare the three inscribed parallelo-hexagons. What relationship is noticeable between them? Why?)

By using the same reasoning, we can now easily extend the result to an octagon ABCDEFGH with the properties that AC //= EG, BD //= FH, CE //= GA and DF //= HB. As shown in Figure 84, inscribed parallelo-octagons are formed if we consecutively connect the midpoints of the sides of such figures. Note however, that in this case the octagons ABCDEFGH are not necessarily parallelo-octagons. Such octagons can easily be drawn by constructing any two parallelograms ACEG and BDFH and then simply connecting A, B, C, D, E, F, G and H.

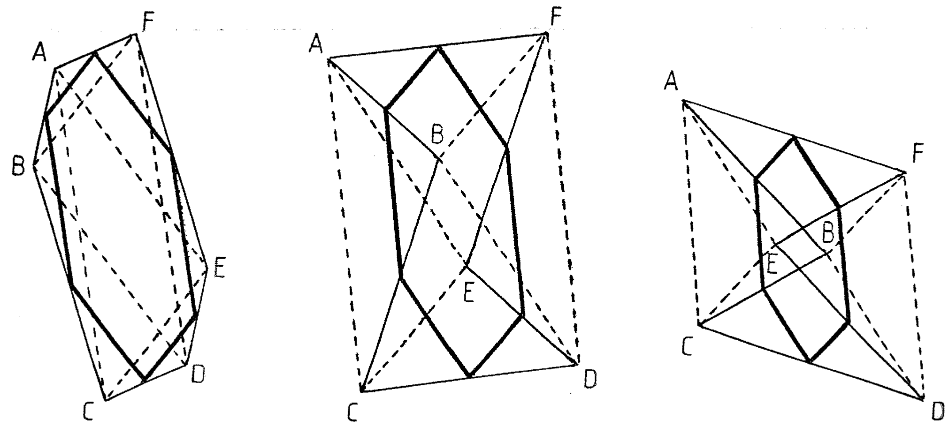

Figure 83

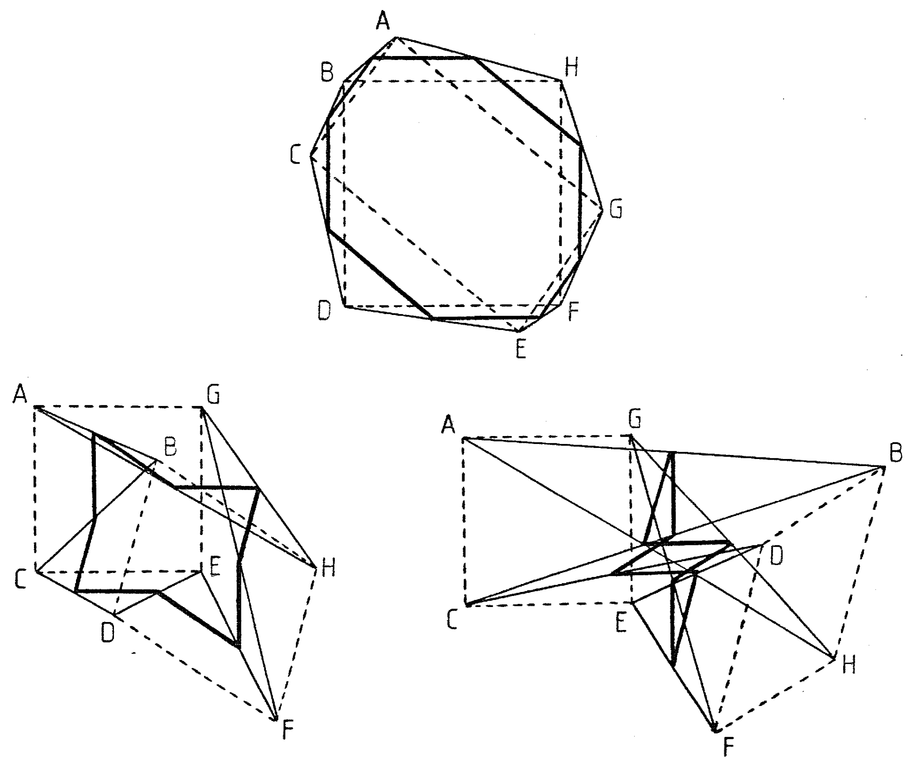

Figure 84

We can now generalize the result as follows:

(1) "If $A_1A_2...A_{2n}$ ($n > 1$) is any $2n$-gon with A_iA_{i+2} //= $A_{i+n}A_{i+n+2}$ ($i = 1; 2; ...n$) and B_j are the midpoints of A_jA_{j+1} ($j = 1; 2; ...2n$), then $B_1B_2...B_{2n}$ is a parallelo-$2n$-gon".

Proof

The proof is straight forward, following directly from the theorem that the line segment connecting the midpoints of two sides of a triangle is parallel and equal to half the length of the third side. For example, as illustrated in Figure 85a where $n = 4$, B_jB_{j+1} //= $\frac{1}{2}A_iA_{i+2}$ and $B_{j+n}B_{j+n+1} = \frac{1}{2}A_{i+n}A_{i+n+2}$. But it is given that A_iA_{i+2} //= $A_{i+n}A_{i+n+2}$ which implies that B_jB_{j+1} //= $B_{j+n}B_{j+n+1}$ and therefore that $B_1B_2...B_{2n}$ is a parallelo-2n-gon.

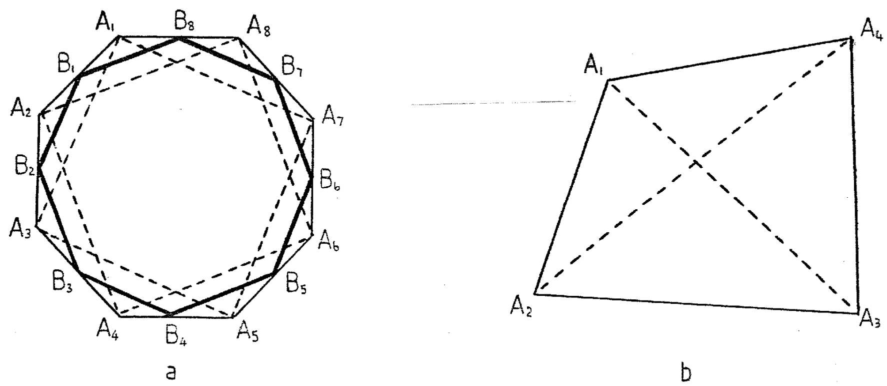

Figure 85

This result is of course further generalizable to 2n-gons with the property A_iA_{i+2}//$A_{i+n}A_{i+n+2}$ or $A_iA_{i+n+2} = A_{i+n}A_{i+n+2}$ for which the respective inscribed 2n-gons would have B_jB_{j+1}// $B_{j+n}B_{j+n+1}$ or $B_jB_{j+1} = B_{j+n}B_{j+n+1}$. (Note that a 2$n$-gon with opposite sides parallel necessarily implies opposite sides equal, and vice versa, only in the case of a parallelogram). Let's also briefly look at the special case of the quadrilaterals (Figure 85b). According to the condition of the above theorem, we should in the case of a quadrilateral $A_1A_2A_3A_4$ have A_1A_3 //= A_3A_1 and A_2A_4 //= A_4A_2, which means that each diagonal must be parallel and equal to itself. But this is obviously true for any orientation or lengths of the diagonals and accounts for the variation of quadrilaterals shown in Figure 81.

The general theorem (1) also has the following interesting corollary, the proof of which is left to the reader:

(2) "The perimeter of the inscribed parallelo-2n-gon is equal to the sum of A_iA_{i+2} (or to half the sum of A_jA_{j+2} where $j = 1; 2;...2n$)".

In the special case of the quadrilaterals, this simply means that the perimeter of the inscribed quadrilateral $B_1B_2B_3B_4$ is equal to the sum of the diagonals of the quadrilateral $A_1A_2A_3A_4$ (compare Denson, 1989).

A second generalization

Let us now return to the original result and critically examine the necessity of B_1, B_2, B_3 and B_4 being midpoints of the sides (see Figure 86). As mentioned before, the result depends on B_1B_2 // A_1A_3 // B_3A_4 and B_1B_4 // A_2A_4 // B_2A_3. How can we maintain these relationships if B_1, B_2, B_3 and B_4 are not midpoints of the sides?

Well, the line segment B_1B_4 would remain parallel to A_2A_4 provided the two sides A_1A_2 and

A_1A_4 are divided in the same proportion (ratio) by B_1 and B_4. Since the same is true for the other line segments, $B_1B_2B_3B_4$ would clearly be a parallelogram if

$$\frac{A_1B_1}{B_1A_2}=\frac{A_1B_4}{B_4A_4}=\frac{A_3B_2}{B_2A_2}=\frac{A_3B_3}{B_3A_4}.$$

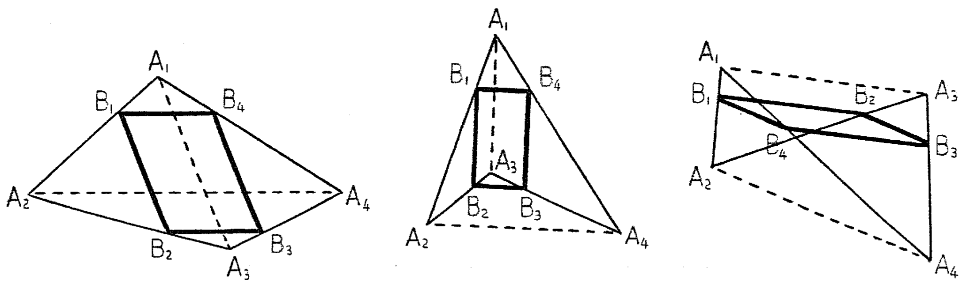

Figure 86

Spontaneously our next questions arise:

(i) What is the relationship between the perimeter of such an inscribed parallelogram and the diagonals of the original quadrilateral? Has it changed or does it stay the same?

(ii) Is this result also generalizable to the $2n$-gons described before?

With regard to the first question, we unfortunately find that the perimeter is no longer necessarily equal to the sum of the diagonals. In fact, if we let

$$\frac{A_1B_1}{B_1A_2}=\frac{A_1B_4}{B_4A_4}=\frac{A_3B_2}{B_2A_2}=\frac{A_3B_3}{B_3A_4}=\frac{p}{q}$$

then it can be easily shown that the perimeter is equal to $2(qA_1A_3 + pA_2A_4)/(p + q)$. Interestingly, if $A_1A_3 = A_2A_4$, this formula reduces to $2A_1A_3$ or $2A_2A_4$.

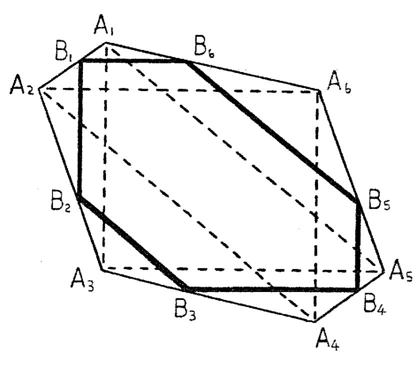

Figure 87

Let us now consider the second question above in relation to the parallelo-hexagon $A_1A_2A_3A_4A_5A_6$ shown in Figure 87. Careful investigation shows that if

$$\frac{A_1B_1}{B_1A_2}=\frac{A_3B_2}{B_2A_2}=\frac{A_3B_3}{B_3A_4}=\frac{A_5B_4}{B_4A_4}=\frac{A_5B_5}{B_5A_6}=\frac{A_1B_6}{B_6A_6}$$

then $B_1B_2B_3B_4B_5B_6$ would clearly be a hexagon with opposite sides parallel. (Note: In Solutions 2, no.17 (Figure 2.31) we called such a hexagon a *parallel-hexagon*, and its generalization to polygons, a *parallel-2n-gon*). In fact, A_iA_{i+2} need not be parallel and equal to $A_{i+3}A_{i+5}$ for this particular result to be true, but only parallel. We can therefore now formulate the following generalization:

(3) "If $A_1A_2...A_{2n}$ ($n > 1$) is any $2n$-gon with $A_iA_{i+2} // A_{i+n}A_{i+n+2}$ (i = 1; 2;...n) and B_j are points on A_jA_{j+1} (j = 1; 2;... $2n$) so that for k = 1; 3; 5;...$2n$ - 1:

$$\frac{A_kB_k}{B_kA_{k+1}} = \frac{A_{k+2}B_{k+1}}{B_{k+1}A_{k+1}} = \frac{p}{q},$$

then $B_1B_2...B_{2n}$ is a parallel-$2n$-gon (opposite sides parallel)."

Proof

The proof follows directly from the theorem that if a line segment connecting two points divides two sides of a triangle into equal ratios then it is parallel to the third side. For example, it implies that $B_jB_{j+1} // A_iA_{i+2}$ and $B_{j+n}B_{j+1+n} // A_{i+n}A_{i+n+2}$. But since $A_iA_{i+2} // A_{i+n}A_{i+n+2}$ we have $B_jB_{j+1} // B_{j+n}B_{j+n+1}$ and therefore that $B_1B_2...B_{2n}$ is a parallel-$2n$-gon.

Of course, in the special case of a quadrilateral we obtain a parallelo-$2n$-gon (parallelogram), since opposite sides parallel implies opposite sides equal. Furthermore, we have the following corollary of theorem (3), proof of which is also left to the reader:

(4) "The perimeter of the inscribed parallel-$2n$-gon is equal to

$$\frac{q\sum_{i=1}^{n} A_{2i-1}A_{2i+1} + p\sum_{i=1}^{n} A_{2i}A_{2i+2}}{p+q}."$$

If we have $\Sigma\, A_{2i-1}A_{2i+1} = \Sigma\, A_{2i}A_{2i+2}$ the perimeter reduces to $\Sigma\, A_{2i-1}A_{2i+1}$ or $\Sigma\, A_{2i}A_{2i+2}$. Of course, if A_jA_{j+2} are all of equal length, let's say d, then the perimeter simply becomes nd.

A counter-example

As mentioned in Chapter 3, quasi-empirical testing/experimentation often plays an important part in the development of a piece of new mathematics. However, it is possibly one of the most neglected aspects when it comes to the teaching of mathematics. Quasi-empirical testing/experimentation is useful since it not only gives us confidence in the validity of our conjectures and theorems, but also an essential concrete understanding/appreciation of their meaning and domains of validity which is sometimes not revealed by logical deduction. More importantly, it can produce counter-examples which necessitate either abandonment or reformulation of conjectures, definitions and/or proofs.

Earlier it was mentioned and proved that *any* hexagon with A_iA_{i+2} //= $A_{i+3}A_{i+5}$ is a parallelo-hexagon. Let's test this observation by trying to construct a hexagon with this property, but different from those shown in Figures 82a, 83 and 87. In the last two cases, the hexagons were drawn by constructing two congruent triangles ACE ($A_1A_3A_5$)and DFB ($A_4A_6A_2$)with AC//DF ($A_1A_3//A_4A_6$), AE//DB ($A_1A_5//A_4A_2$) and CE//FB ($A_3A_5//A_6A_2$), and then connecting A(A_1), B(A_2), C(A_3), D(A_4), E(A_5) and F(A_6). Or in other words, by first constructing ΔACE ($A_1A_3A_5$) and then rotating it through 180° to map onto ΔDFB($A_4A_6A_2$). However, we can also maintain the parallelness of A_iA_{i+2} and $A_{i+3}A_{i+5}$ by translating $\Delta A_1A_3A_5$ to map onto $\Delta A_4A_6A_2$, and then connecting the vertices to obtain the figures shown in Figure 88.

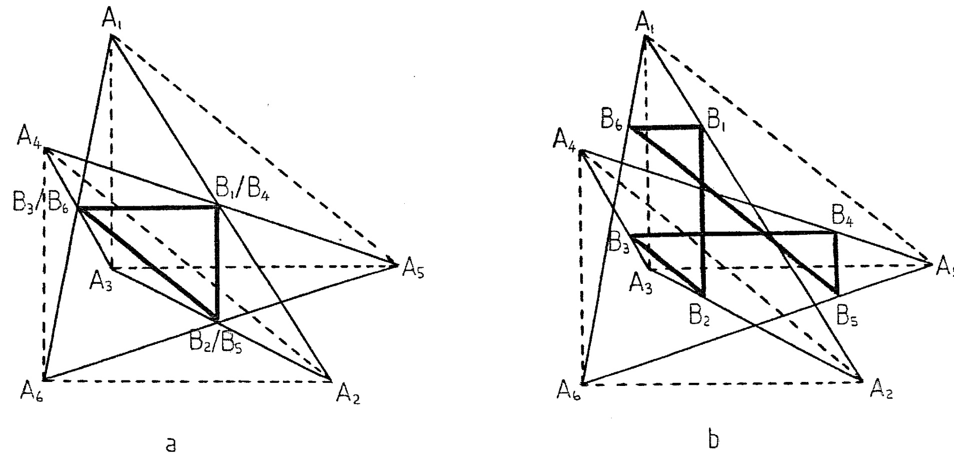

Figure 88

These figures are clearly not parallelo-hexagons as opposite sides are not parallel and equal, but in fact are unequal in length and intersect. In this case, the earlier given proof is false since it is based on the implicit assumption that the opposite sides A_iA_{i+1} and $A_{i+3}A_{i+4}$ do not intersect, and is therefore applicable only to the type of configuration shown in Figures 82a, 83 and 87. However, this example does not invalidate generalizations (1) and (3), as illustrated respectively by Figures 88a and 88b. Also note that in the first case opposite sides B_iB_{i+1} and $B_{i+3}B_{i+4}$ actually coincide to produce a degenerate parallelo-hexagon in the form of a triangle. (This happens because the opposite sides of A_1A_2... A_6 intersect in their midpoints).

Considering converses

Let's now consider possible converses of Theorems 1 and 3. In both these theorems we connected points which divided the sides of a $2n$ -gon proportionally, to obtain another $2n$ -gon with opposite sides equal and parallel, or just parallel. Conversely, we should therefore now consider the case where we start at the midpoints, or any other point B_1 of a particular side, say A_1A_2 and drawing $B_1B_2//A_1A_3$, $B_2B_3//A_2A_4$, etc. with B_2, B_3, etc. on A_2A_3, A_3A_4, etc.

For example, what happens if we start with the midpoint B_1 of side A_1A_2 of the hexagon with $A_iA_{i+2}//= A_{i+3}A_{i+5}$ shown in Figure 89a, and we draw $B_jB_{j+1}//A_jA_{j+2}$? Similarly, what happens if we start out with any point B_1 of side A_1A_2 of the hexagon with

$A_iA_{i+2}//=A_{i+3}A_{i+5}$ shown in Figure 89b, and we draw $B_jB_{j+1}//A_jA_{j+2}$? What happens if A_iA_{i+2} is only parallel to $A_{i+3}A_{i+5}$, but not equal? What happens if we choose B_1 on any of the other sides?

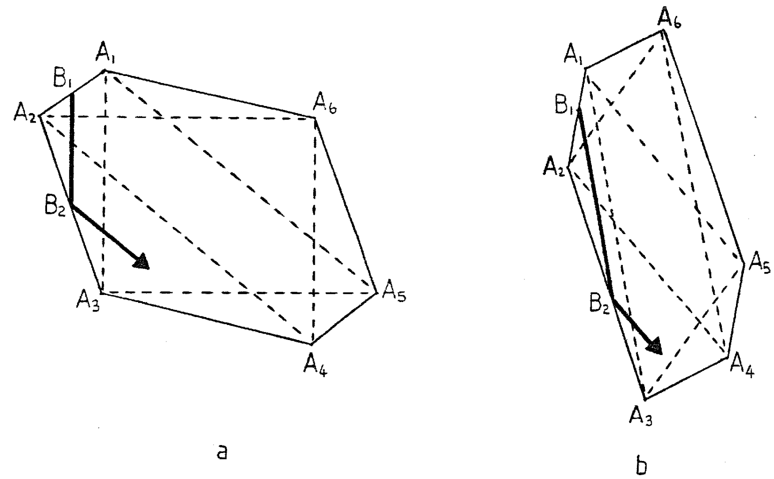

Figure 89

Complete the figures above by continuing to draw $B_jB_{j+1}//A_jA_{j+2}$. What do you notice? Also construct hexagons with A_iA_{i+2} only parallel to $A_{i+3}A_{i+5}$, but not equal, and repeat the previous exercise with B_1 a midpoint or any other point. What do you notice? Can you formulate and explain your observations?

Although intuitively one may anticipate that one could go on indefinitely without necessarily returning to the original starting point, we find the rather surprising result that we always return to the start no matter where the starting points are chosen. Furthermore, in the two figures above we always find a parallelo-hexagon, while in the second case with A_iA_{i+2} only parallel to $A_{i+3}A_{i+5}$, always a parallel-hexagon. These observations can now be generalized to the following corresponding converses to Theorems 1 and 3:

(5) "If $A_1A_2 \ldots A_{2n} (n > 1)$ is any $2n$ - gon with $A_iA_{i+2}//= A_{i+n}A_{i+n+2}$ ($i = 1; 2; \ldots n$) and B_jB_{j+1} is drawn parallel to A_jA_{j+2} ($j = 1; 2; \ldots 2n$) starting from the midpoint B_1 of A_jA_{j+1}, then a *closed* parallelo-$2n$-gon $B_1B_2 \ldots B_{2n}$ is formed"

(6) "If $A_1A_2 \ldots A_{2n} (n > 1)$ is any $2n$-gon with $A_iA_{i+2}//A_{i+n}A_{i+n+2}$ ($i = 1; 2; \ldots n$) and B_jB_{j+1} is drawn parallel to A_jA_{j+2} ($j = 1; 2; \ldots 2n$) starting from any point B_1 on a side A_jA_{j+1}, then a *closed* parallel-$2n$ - gon $B_1B_2 \ldots B_{2n}$ is formed."

Proof

We shall now only prove that the 2n-gons referred to above are closed, leaving the rest of the two proofs to the reader to complete. Consider the case where B_1 is on A_1A_2. It is therefore required for us to prove that the point B_{2n+1} is the same as B_1. Since B_jB_{j+1} is drawn parallel to A_jA_{j+2} we have as before:

$$\frac{A_1B_1}{B_1A_2} = \frac{A_3B_2}{B_2A_2} = \frac{A_3B_3}{B_3A_4} = \frac{A_5B_4}{B_4A_4} = \ldots \text{ etc., or in general:}$$

$$\frac{A_kB_k}{B_kA_{k+1}} = \frac{A_{k+2}B_{k+1}}{B_{k+1}A_{k+1}} = \frac{p}{q} \quad \text{where } k = 1; 3; 5;\ldots$$

Considering $k = 2n + 1$ we will therefore have that

$$\frac{A_1B_1}{B_1A_2} = \frac{A_3B_2}{B_2A_2} = \ldots = \frac{A_{2n+1}B_{2n+1}}{B_{2n+1}A_{2n+2}} = \frac{A_{2n+3}B_{2n+2}}{B_{2n+2}A_{2n+2}}.$$

Compare the first and third ratios above. Since A_{2n+1} and A_{2n+2} are the same points as A_1 and A_2 respectively, we therefore have that the point B_{2n+1} and B_1 are the same point. Similarly, if we choose B_1 on any of the other sides, we can show that B_1 and B_{2n+1} are identical.

Looking back

However, looking back and reflecting on our preceding proof, it should be clear that we did not at all utilize the property $A_iA_{i+2}//A_{i+n}A_{i+n+2}$. This proof therefore immediately provides us with the following interesting generalization (illustrated for a hexagon in Figure 90):

(7) "If $A_1A_2\ldots A_2(n>1)$ is any $2n$-gon and B_jB_{j+1} is drawn parallel to A_jA_{j+2} ($j = 1; 2;\ldots 2n$) starting from any point B_1 on a side A_jA_{j+1}, then a *closed* $2n$-gon $B_1B_2\ldots B_{2n}$ is formed".

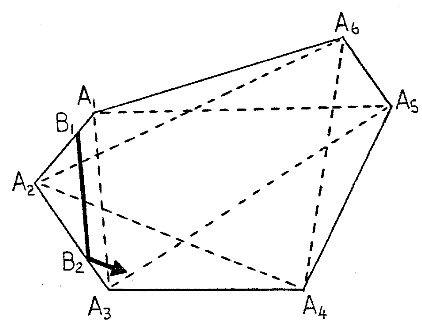

Figure 90

(It is left to the reader to complete the figure in Figure 90).

An analogous result

Let's now examine what happens if we carry out the same procedure of drawing $B_jB_{j+1}//A_jA_{j+2}$ on a triangle, pentagon, septagon, etc. The reader is now invited to carry out this procedure on the two figures shown in Figure 91. What do you notice? Can you form a generalization?

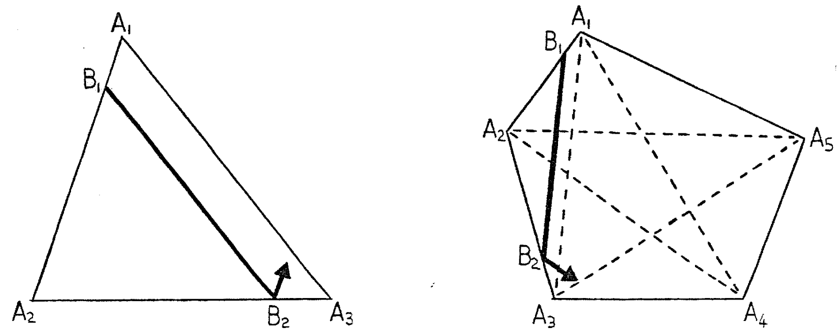

Figure 91

Investigating some cases the following generalization analogous to Theorem 7 can be made:

(8) "If $A_1A_2...A_{2n-1}$(n >1) is any (2n-1)-gon and B_jB_{j+1} is drawn parallel to A_jA_{j+2}(j = 1; 2;...4n-2) starting from any point B_1 on a side A_jA_{j+1}, then a *closed* (4n-2)-gon $B_1B_2...B_{4n-2}$ is formed".

Proof

The proof is straight forward, for example, consider the case where B_1 is on A_1A_2. Let us now first consider the point B_{2n} which will also be on A_1A_2. Since B_jB_{j+1} is drawn parallel to A_jA_{j+2} we have as before:

$$\frac{A_kB_k}{B_kA_{k+1}} = \frac{A_{k+2}B_{k+1}}{B_{k+1}A_{k+1}} = \frac{p}{q} \text{ where } k = 1; 2; 3; 5;...$$

Considering $k = 2n$ -1 we will have that :

$$\frac{A_1B_1}{B_1A_2} = ... = \frac{A_{2n-1}B_{2n-1}}{B_{2n-1}A_{2n}} = \frac{A_{2n+1}B_{2n}}{B_{2n}A_{2n}}.$$

Compare the first and third ratios above. Since A_{2n+1} and A_{2n} are the same points as A_2 and A_1 respectively, it clearly follows that B_1 and B_{2n} will be the same point only when B_1 is the midpoint.

Let us now show that B_{4n-1} is the same point as B_1. Considering $k = 4n$ -1 we will have that:

$$\frac{A_1B_1}{B_1A_2} = ... = \frac{A_{4n-1}B_{4n-1}}{B_{4n-1}A_{4n}} = \frac{A_{4n+1}B_{4n}}{B_{4n}A_{4n}}.$$

Compare the first and second ratios above. Since A_{4n-1} and A_{4n} are the same points as A_1 and A_2 respectively, we therefore have that the point B_{4n-1} and B_1 are the same point. Similarly, if we choose B_1 on any of the other sides, we can show that B_1 and B_{4n-1} are identical.

Some reflections

The generalizations described in this chapter would probably not have been possible to make simply by blind, trial and error experimentation, since an understanding of the *explanatory* property which makes it true for quadrilaterals, proved indispensable throughout the whole exploration. In particular, the generalization from Theorems (5) and (6) to Theorem 7 is a good example of the *discovery* function of proof described in Chapter 3, whereby the deductive indentification of the explanatory property of a particular result often enables further generalization.

It is furthermore important to distinguish between two different kinds of generalization demonstrated here, namely *inductive* generalization and *deductive* generalization. With inductive generalization is meant here that a generalization is initially made on quasi-empirical grounds without necessarily any deductive thought involved, for example observing and formulating generalizations like Theorems 5 and 6 from the consideration of some particular cases like the figures in Figure 9. A deductive generalization on the other hand is made on the basis of a logical deduction, for example by deductively analysing the conditions of a particular theorem (or theorems) and finding from its proof that a specific condition is sufficient, but not necessary, thereby enabling further generalization. The generalization of Theorem 7 from the proofs of Theorems 5 and 6 is therefore also a good example of deductive generalization.

Further questions

In keeping with the spirit of this book, inquisitive readers may wish to follow up with questions like the following for further exploration, or add questions of their own:

(a) Can you find analogous results of the preceding theorems involving equal-2n-gons? (Opposite sides equal - see Solutions 2, no.17).

(b) (i) Can you draw a quadrilateral configuration to give a degenerate parallelogram when the midpoints of the sides are connected? What figure can be anticipated?

(ii) Can you draw an octagon configuration to give a degenerate parallelo-octagon when the midpoints of the sides are connected? What figure can be anticipated?

(c) Specialization is often also a useful problem posing strategy. For example, consider the following two special cases of the original theorem (see Figure 53):

(i) If the midpoints of the sides of any perpendicular quad are connected, then we obtain a rectangle. Can you generalize this result?

(ii) If the midpoints of the sides of any diagonal quad are connected, then we obtain a rhombus. Can you generalize this result?

(iii) Further specialization would be to ask under what conditions would we obtain a square and to try and generalize it.

(d) A further interesting property of the original theorem for convex quadrilaterals is that the area of the inscribed parallelogram EFGH is half the area of the quadrilateral ABCD (see Figure 81). (Hint: compare the area of ΔAEH with that of ΔABD, the area of ΔBFE with that of ΔBCA, etc.).

(i) Is this result true for concave and crossed quadrilaterals?

(ii) Can you generalize it?

(e) Consider the case shown in Figure 92 where the midpoints of the sides of a triangle have been connected. Does it have any properties which are generalizable to $(2n-1)$-gons? Can you extend the result to other points on the sides of a triangle and generalize to $(2n-1)$-gons?

(f) Consider the regular hexagon (star of David) shown in Figure 93. If the midpoints of the sides are connected as shown, another star of David is obtained. Can you generalize this result?

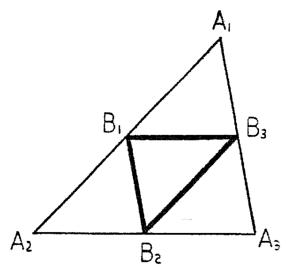

Figure 92

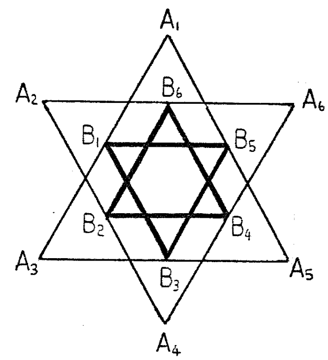

Figure 93

(g) Another result directly related to the original theorem is given in Coxeter & Greitzer (1967:54), namely: the segments B_1B_3 and B_2B_4 joining the midpoints of the opposite sides of a quadrilateral and the segment C_1C_2 joining the midpoints of the diagonals are concurrent and bisect one another (see Figure 94). Can you also generalize this result?

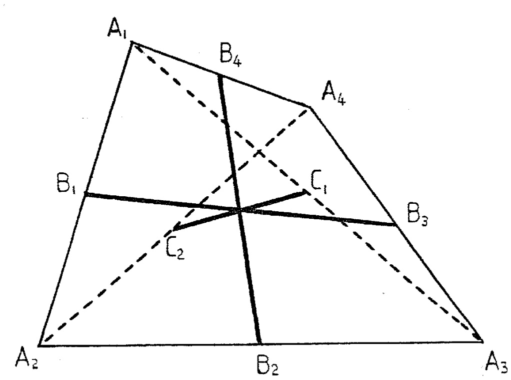

Figure 94

(h) We can specialize the original theorem in another way. For example, if sides A_1A_2 and A_2A_3 are lying in a straight line as shown in Figure 95 and the midpoints are connected as before, an inscribed parallelogram is obtained in the triangle $A_1A_3A_4$. We can therefore inscribe a parallelogram in any triangle by consecutively connecting the midpoints of two sides with the midpoints of any two subdividing sections of the third side (in which case the one side of the parallelogram coincides with part of the third side and is half its length). Can you generalize this result?

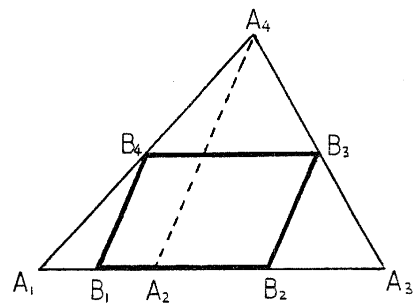

Figure 95

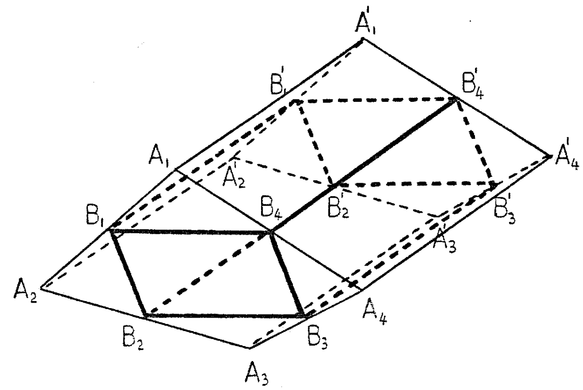

Figure 96

(i) Suppose we translate a quadrilateral $A_1A_2A_3A_4$ with an inscribed Varignon parallelogram $B_1B_2B_3B_4$ some distance into three dimensional space to obtain their images $A'_1A'_2A'_3A'_4$ and $B'_1B'_2B'_3B'_4$, and consider the three dimensional figures traced out by the translation of the vertices. As shown in Figure 96 we would then have a *parallelo-piped* (a solid bounded by six parallelograms as its faces, the opposite pairs being congruent and parallel) inscribed in a *prism* (a solid with two parallel congruent polygons as opposite faces with edges joining corresponding vertices so that the remaining faces are parallelograms). Can you generalize this result as before? Can you apply the theorems in this chapter, as well as the area relationship mentioned in question (d) above? Can you further generalize to *n* dimensions?