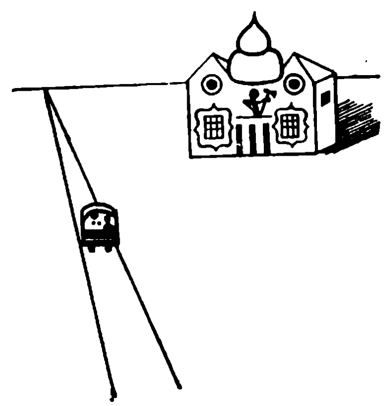
5 Level Curves

From Vasilyev, N.B. & Gutenmacher, V.L. (1980).
Straight Lines & Curves. Mir Publishers, Moscow.

In this section the problems and the theorems of the previous section are discussed, using a new terminology. The concepts we are going to meet in this section are *functions defined on a plane and their level curves*. These are useful especially in the solutions of the maximum and minimum problems.

The "Bus" Problem

5.1. A tourist bus is travelling along a straight highway. A palace is situated by the side of the highway, at some angle to the highway. At what point on the highway should the bus stop for the tourists to be able to see the facade of the palace from the bus in the best possible way?



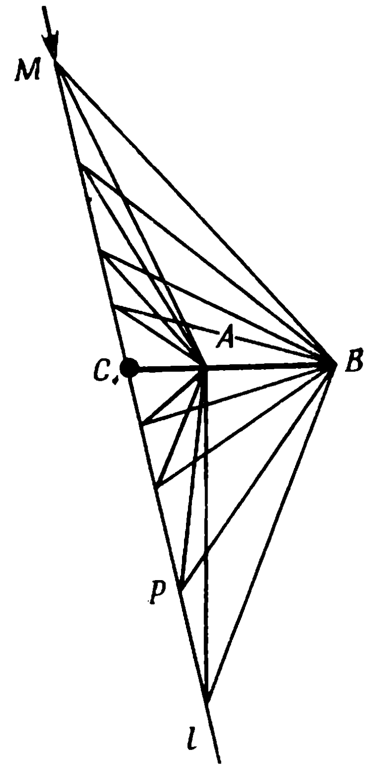
Mathematically the problem may be formulated as follows.

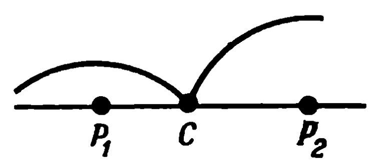
A straight line l and a segment AB, which does not intersect it are given. Find on the straight line l a point P

for which the angle APB assumes its maximum value.

Let us first have a look at how the angle AMB changes, when the point M moves along the straight line l. In other words, let us look at the behaviour of the function f which relates each point M of the line to the size of the corresponding angle $\widehat{AMB}$.

It is easy to draw a rough graph of this function. (Remember that a graph is drawn in the following way: above each point M of our straight line a point at a distance of $f(M) = \widehat{AMB}$ is plotted.)

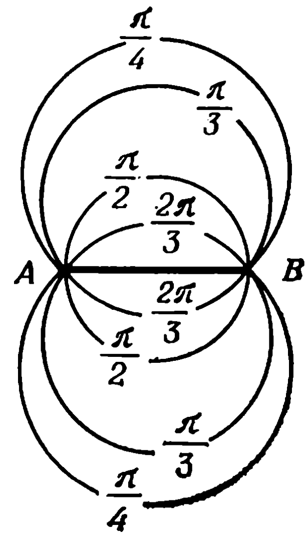




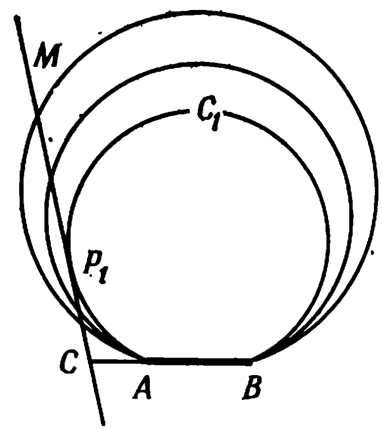
The problem may be solved analytically: introduce coordinates on the straight line l, express the value of the angle AMB in terms of the x-coordinate of the point M and find for what value of x, the function obtained reaches its maximum. However, the formula for $f(x)$ is quite complicated.

We shall give a more elementary and instructive solution. But to do this we have to study how the value of the angle AMB depends on the position of the point M in the whole plane (and not only on the straight line l).

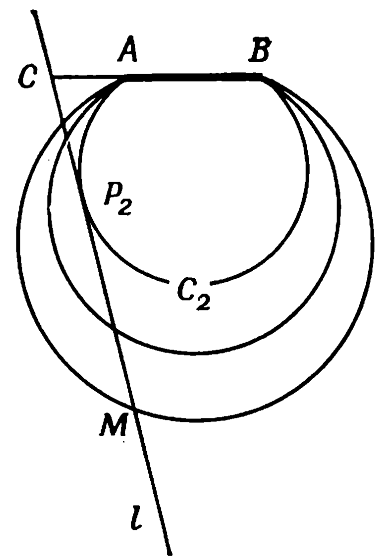
□ A set of points M in the plane, for which the angle AMB assumes a given value φ is a pair of symmetric arcs with their end-points at A and B (see Sec. 2, proposition **E**). If these arcs are drawn for different values of φ (where $0 < \varphi < \pi$), we get a family of arcs which cover the whole plane except the straight line AB. In the figure a few of these arcs are drawn and on each of them is marked its corresponding value of φ. For example, a circle with diameter AB corresponds to the value $\varphi = \pi/2$.



We shall now consider only the points M on the straight line l. From them we have to select that point for which the angle AMB assumes its maximum value. Through each point there passes some arc of our family: if $\widehat{AMB} = \varphi$, the point M lies on the arc corresponding to the value φ. Thus, the problem is reduced to the following: from all the arcs crossing the line l, select the one which corresponds to the maximum value of $\widehat{AMB} = \varphi$.

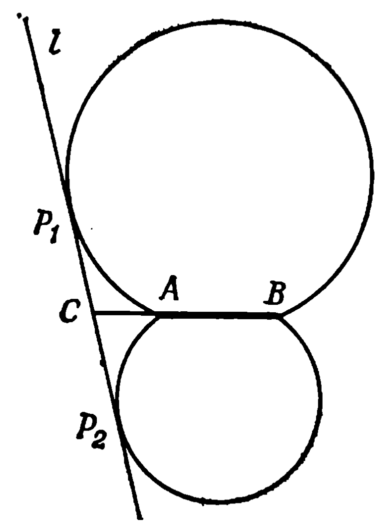


We shall examine the part of the straight line l located to one side of the point C, the point of intersection of the straight line AB with l. (We shall not consider the case, when the segment AB is parallel to the line l—we leave that to the reader.) We shall draw the arc c_1 touching this part of the straight line and prove that the segment AB subtends the maximum angle at the point of tangency P_1. Any point M of the straight line l, except P_1, lies outside the segment cut off by the arc c_1. As we know (proposition E, page 40), from this it follows that $\widehat{AMB} < \widehat{AP_1B}$.



It is obvious that on the other side of the point C everything will be exactly the same: the point P_2, at which the angle subtended by the segment AB is a maximum, is also the point of tangency of the straight line with one of the arcs of our family.

We have thus proved that the required point P of our problem coincides with one of the points P_1 or P_2 at which the circles passing through the points A and B touch the straight line l.



We should select as P the point for which the angle PCA is an acute angle. If the segment AB is perpendicular to the line l, then from symmetry considerations it is immediately obvious that the points P_1 and P_2 are completely equivalent; hence the number of points, solving the problem, in this case is two. (However the tourists, in any case must select that point P_1 or P_2 from which the facade of the palace is visible.)

Functions on a Plane. The main idea of the solution of problem 5.1 is to investigate over the whole plane