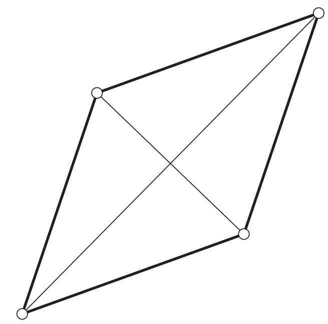
Systematizing Rhombus Properties

You have earlier discovered or learned that, among others, a rhombus has the following properties:

- All sides are equal.
- The diagonals are perpendicular.
- The diagonals bisect each other.
- There are two axes of symmetry (through the two pairs of opposite angles).
- Opposite sides are parallel.



DESCRIBE

How would you describe what a rhombus is, over the telephone, to someone who is not yet acquainted with a rhombus?

1. Which of the following descriptions do you think you would be able to use? Circle these descriptions.

a. A rhombus is any quadrilateral with opposite sides parallel.

b. A rhombus is any quadrilateral with perpendicular diagonals.

c. A rhombus is any quadrilateral with two perpendicular axes of symmetry (each through a pair of opposite angles).

d. A rhombus is any quadrilateral with perpendicular, bisecting diagonals.

e. A rhombus is any quadrilateral with two pairs of adjacent sides equal.

f. A rhombus is any quadrilateral with all sides equal.

g. A rhombus is any quadrilateral with one pair of adjacent sides equal, and opposite sides parallel.

One way of testing a description is to construct a figure complying with the description to see if it really gives the desired figure.

2. Open the sketch **Rhombus.gsp** and check each of the descriptions a–g on page 133. Press each button step by step on each of the seven pages to construct the figures. When each construction is finished, match each page with a description in the table. Drag each figure to see if it always remains a rhombus. (*Note:* Since a rhombus can be dragged into the shape of a square, we regard a square as a special rhombus.) In the table, cross out the names of any pages that construct quadrilaterals that are not always rhombuses.

Page	Description (a–g)
Rhombus 1	
Rhombus 2	
Rhombus 3	
Rhombus 4	
Rhombus 5	
Rhombus 6	
Rhombus 7	

3. List the descriptions from a–g that you think correctly describe a rhombus.

4. State the description from a–g that you personally think best describes a rhombus. Try to defend your choice with good reasons.

5. Carefully examine the following descriptions and comment on their suitability.

a. A rhombus is any quadrilateral with equal diagonals.

b. A rhombus is any quadrilateral with all sides equal, opposite sides parallel and perpendicular, and bisecting diagonals.

c. A rhombus is any quadrilateral that looks like a rhombus.

d. A rhombus is any quadrilateral with all sides, but not all angles, equal.

CHALLENGE Using only logical deduction, can you prove that all of the five properties of a rhombus listed at the beginning and not included in your descriptions in Question 2 can be derived from them? Start from the description as your given assumption and then prove as theorems that a rhombus has each of the other properties listed at the beginning. Apart from using your description as an assumption in these proofs, you can use any new theorems that you prove in the subsequent proofs of the other properties.

PROVING RHOMBUS PROPERTIES FROM DEFINITIONS

> *When we look at the history of mathematics, we see a kind of lifelike, elemental rhythm. There are periods of exuberant untidy growth, when exciting, vital structures rise upon untried assumptions, and loose ends lie about all over the place. Logic and precision are not unduly honored; because restlessness, enthusiasm, daring and ability to tolerate a measure of confusion are the appropriate qualities of mind at these times. Such periods are followed by pauses for consolidation, when the analysts and systematizers get to work; material is logically ordered, gaps are filled, loose ends are neatly tied up, and rigorous proofs are supplied. Solemn commentators sit in judgment upon great innovators. Whole areas of mathematics are formed into deductive systems, based on sets of unproved, explicitly stated axioms.*
>
> —*L. W. H. Hull, 1969*

We will concern ourselves here with the second part of the quotation above, namely, a logical organization of the properties of a rhombus. The function or purpose of proof here will therefore not be the explanation, discovery, or verification of the properties of a rhombus, but their systematization.

In the preceding part of this activity, you found that each of the following descriptions could be used to accurately construct a rhombus:

A. A rhombus is any quadrilateral with two perpendicular axes of symmetry (each through a pair of opposite angles).

B. A rhombus is any quadrilateral with perpendicular, bisecting diagonals.

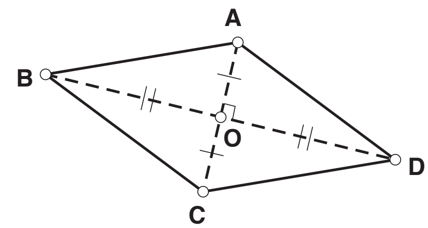
C. A rhombus is any quadrilateral with all sides equal.

D. A rhombus is any quadrilateral with one pair of adjacent sides equal and opposite sides parallel.

In mathematics, we call such descriptions *definitions*. As we can see, there may be many different, alternative ways in which we can define mathematical objects. We now have to show that all the other properties of a rhombus logically follow as theorems from each of these definitions. We will now give an example for definition B.

Definition: A rhombus is any quadrilateral with perpendicular, bisecting diagonals.

Consider the figure where a quadrilateral is given, with diagonals *AC* and *BD* perpendicularly bisecting each other at *O*.



Theorem 1: All sides of a rhombus are equal.

6. What can you say about triangles *ABO* and *ADO*? Why?

7. From Question 6, what can you conclude about sides *AB* and *AD*?

8. What can you say about triangles *ABO* and *CBO*? Why?

9. From Question 8, what can you conclude about sides *AB, CB,* and *AD*?

10. What can you say about triangles *ADO* and *CDO*? Why?

11. From Question 10, what can you now conclude about all four sides *AD, CD, AB,* and *CB*?

Theorem 2: The diagonals of a rhombus bisect the pairs of opposite angles.

12. What can you say about triangles *ABC* and *ADC*? Why?

13. From Question 12, what can you conclude about angles *BAC* and *DAC,* as well as angles *BCA* and *DCA*?

14. What can you say about triangles *ABD* and *CBD*? Why?

15. From Question 14, what can you conclude about angles *ABD* and *CBD,* as well as angles *ADB* and *CDB*?

Theorem 3: The diagonals of a rhombus are axes of symmetry.

16. From Question 12 in the previous proof, what can you conclude about line *AC*? Why?

17. From Question 14 in the previous proof, what can you conclude about line *BD*? Why?

Theorem 4: The opposite sides of a rhombus are parallel.

18. What can you say about triangles *ABO* and *CDO*? Why?

19. From Question 18, what can you conclude about angle *BAO* and angle *DCO*?

20. From Question 19, what can you now conclude about sides *AB* and *CD*?

21. Use the same argument as in Questions 18–20 to complete the proof for the remaining two sides.

Present Your Proofs

Write out your proofs clearly for presentation to your group or class.

Further Exploration

1. Now choose any two of the other three possible definitions A, C, and D for a rhombus. For each, show, as in the example on the previous page, how the remaining properties listed at the beginning and not included in your definition can be proved as theorems.
2. A concept can also be defined in terms of its relationships with other concepts. A rhombus can also be viewed as a special parallelogram or a special kite, since both of these can be dragged into the shape of a rhombus. Try to define a rhombus by making use of these relationships.
3. A rhombus can also be viewed as a special circum quadrilateral (that is, a quadrilateral circumscribed around a circle). Try to define a rhombus as a circum quadrilateral with additional properties.

Class Discussion

A definition can be seen as an agreement among interested parties about what a specific object is. Although you have now seen that it is possible to define a rhombus in many different ways, it can be very confusing if everyone is using a different definition. It is therefore now necessary to choose a common definition that will be acceptable for the whole class. Have a class discussion to decide which definition of a rhombus is most convenient for you.

REASONING BACKWARD: PARALLEL LINES
(PAGE 131)

This worksheet also focuses on the systematization function of proof, since we are proving a result here that was used earlier to prove another result. If you have not yet done so, read the Teacher Notes for the Reasoning Backward: Triangle Midpoints activity.

Prerequisites: Knowledge of the AA condition of similarity and the algebra of ratios.

Sketch: No sketch is required for this activity. If students wish to reinvestigate this theorem, they can use the sketch **Parallel.gsp.**

PROVING

1. Angle ADE = angle ABC, since they are corresponding and $\overline{DE} \parallel \overline{BC}$.
2. Triangle ADE is similar to triangle ABC (AA).
3. $\frac{AB}{AD} = \frac{AC}{AE}$.
4. $\frac{AD + DB}{AD} = \frac{AE + EC}{AE}$.
5. $\frac{AD + DB}{AD} - \frac{AD}{AD} = \frac{AE + EC}{AE} - \frac{AE}{AE} \longrightarrow \frac{DB}{AD} = \frac{EC}{AE}$.
6. If D is the midpoint of $\overline{AB}$, E will also be the midpoint of $\overline{AC}$. The converse of the triangle midpoint theorem is therefore a special case of this theorem. Similarly, the triangle midpoint theorem itself is a special case of the converse of this theorem (see below).

Further Exploration

If two sides of a triangle are divided in the same ratios by two points, then a line through those two points will be parallel to the third side.

Although the proof is similar to the previous one (but in reverse order), some students may need your help. The proof follows the answers to Questions 5, 4, and 3, in that order, to show that the triangles are similar by SAS similarity. Conclude, therefore, that corresponding angles ADE and ABC are equal, and hence $\overline{DE} \parallel \overline{BC}$.

SYSTEMATIZING RHOMBUS PROPERTIES
(PAGE 133)

The main purpose of this activity is to introduce students to the systematization function of proof: the fact that proof is an indispensable tool in the organization of known results into a deductive system of definitions and theorems. Students should know the properties of a rhombus well. It should be made clear to students that the main objective of these worksheets is not to determine whether these properties are true or not, but to investigate their underlying logical relationships, as well as different possible systematizations. However, an element of verification is present, in the sense that the given definitions have to be logically evaluated to see whether all the other properties not included in the definition can be derived from it.

Further objectives are

- Developing students' understanding of the nature of definitions as unproved assumptions, as well as the existence of alternative definitions.
- Engaging students in the evaluation and selection of different formal, economical definitions rather than just providing them with a single ready-made definition.
- Developing students' ability to construct formal, economical definitions for geometrical concepts.

For a more detailed discussion of defining as a mathematical activity and where it fits into the van Hiele theory, read the discussion in the Teacher Notes for the Systematizing Isosceles Trapezoid activity.

Prerequisites: Knowledge of the properties of a rhombus, parallel lines, and conditions for congruency.

Sketch: Rhombus.gsp.

DESCRIBE

The purpose of this activity is to introduce students to a mathematical definition as an economical but accurate description of an object.

1. Responses may vary.

2.

Sketch page	Desc. (a–g)	Comments
~~*Rhombus 1*~~	b	Point out that this condition is necessary but not sufficient.
Rhombus 2	d	This sketch and description are correct.
Rhombus 3	c	The sketch is correct. Strictly speaking, it is redundant to state that the two axes of symmetry have to be perpendicular, since it can be proven that if a figure has only two axes of symmetry, they are perpendicular to each other.
~~*Rhombus 4*~~	a	This description is a necessary, but not sufficient, condition.
Rhombus 5	f	The sketch is correct, and the description is a correct definition.
~~*Rhombus 6*~~	e	This sketch constructs a kite. The description is of a necessary, but not sufficient, condition.
Rhombus 7	g	The sketch and description are correct.

3. c, d, f, and g.
4. Answers will vary, although f is the most economical definion.
5. a. This description is wrong because it contains an *incorrect* property, since rhombuses do not (in general) have equal diagonals.

   b. This one is correct, but *uneconomical* (i.e., it contains more information than is necessary).

   c. This one is *circular;* it is completely unacceptable to define an object in terms of itself, because that does not explain what the object is.

   d. This description does not allow the inclusion of the squares as special cases of rhombuses. Although it is not mathematically incorrect to describe a rhombus in this way, it is *not convenient* to do so. First, a *partition* description (definition) such as this is always longer than an *inclusive* one (because of having to add qualifiers such as "not all angles equal"). Second, a partition description (definition) invariably increases the number of theorems we have to prove in a deductive system (for example, we have to prove separately that the diagonals of a square bisect each other perpendicularly, instead of just assuming it from an inclusive view in which it is seen as a special rhombus).

PROVING RHOMBUS PROPERTIES FROM DEFINITIONS

Point out that from the given definition, deductive orderings other than the one given below are possible.

6. They are congruent (SAS).
7. $AB = AD$.
8. They are congruent (SAS).
9. $AB = CB = AD$.
10. They are congruent (SAS).
11. $AD = CD = AB = CB$.
12. They are congruent (SSS). From theorem 1, $AB = AD$, $CB = CD$, and AC is common.
13. Angle BAC = angle DAC, and angle BCA = angle DCA.
14. They are congruent (SSS).
15. Angle ABD = angle CBD, and angle ADB = angle CDB.
16. Line AC is an axis of symmetry, since a reflection of triangle ABC around AC maps it onto ADC.
17. Line BD is an axis of symmetry, since a reflection of triangle ABD around BD maps it onto CBD.

    Note: If a rhombus is regarded as a special parallelogram, theorem 4 and its proof are redundant. However, a proof is given simply to show that can it be derived from the given definition.
18. They are congruent (SAS).
19. Angle BAO = angle DCO.
20. $\overline{AB} \parallel \overline{CD}$, since the alternate angles BAO and DCO are equal.
21. The argument is similar.

Further Exploration

1. Responses will vary.
2. Several different possibilities exist; for example:
   a. A rhombus is a parallelogram with one pair of adjacent sides equal (equivalent to g, on the previous page).
   b. A rhombus is a parallelogram with perpendicular diagonals.
   c. A rhombus is a parallelogram with a diagonal bisecting one of its angles.
   d. A rhombus is a kite with one pair of opposite sides parallel.
   e. A rhombus is a kite with three angles bisected by its diagonals.
3. Several different possibilities exist; for example:
   a. A rhombus is a circum quadrilateral with three equal sides.
   b. A rhombus is a circum quadrilateral with opposite sides parallel.
   c. A rhombus is a circum quadrilateral with bisecting diagonals.

   (*Hint:* In all these examples, use the property of a circum quadrilateral that the two sums of its opposite sides are equal.)

   d. A rhombus is a circum quadrilateral with its diagonals intersecting at its incenter.

Class Discussion

A good definition of a concept is one that allows us to easily deduce the other properties of the concept; that is, it should be *deductive-economical.* It might be a good exercise for students to compare different definitions according to this criterion. For example, the definition of a rhombus as a quadrilateral with two axes of symmetry through the opposite angles is more deductive-economical than the standard textbook definition of it as a quadrilateral with all sides equal. For example, for the former, the other properties (e.g., perpendicular, bisecting diagonals, all sides equal, etc.) follow immediately from symmetry, whereas with the latter, we have to use congruency and somewhat longer arguments to deduce the other properties.

Another way in which we could compare different definitions is to see whether or not a particular definition allows us to directly construct the object being defined. For example, defining a rhombus as any quadrilateral with one pair of adjacent sides equal and opposite sides parallel allows us to construct it easily. However, defining it as a circum quadrilateral with diagonals bisecting at its incenter (although this is valid as a definition) does not allow us to construct it directly from the properties given in the definition. The former definition could be called a *constructable* definition, whereas the latter could be called a *nonconstructable* definition. It is customary (although this is not always done) to choose constructable definitions in mathematics.