The Tangential or Circumscribed Quadrilateral

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Introduction

Although a substantial part of the South African school geometry curriculum focuses on exploring the properties of cyclic quadrilaterals, the study of quadrilaterals circumscribed around a circle is somewhat neglected. This is a pity, as the mathematical content falls well within reach of high school learners and utilises no geometry theorems that are not already in the curriculum. Moreover, having learners investigate such quadrilaterals provides opportunities for reinforcement of many previously learnt concepts.

A tangential or circumscribed quadrilateral can be defined simply as a quadrilateral that has an incircle (its sides therefore being tangents to the incircle). This brings us to the first easy to prove theorem involving angle bisectors.

Angle Bisectors

Theorem 1: The angle bisectors of any tangential quadrilateral are concurrent at the incentre of the quadrilateral (see Figure 1).

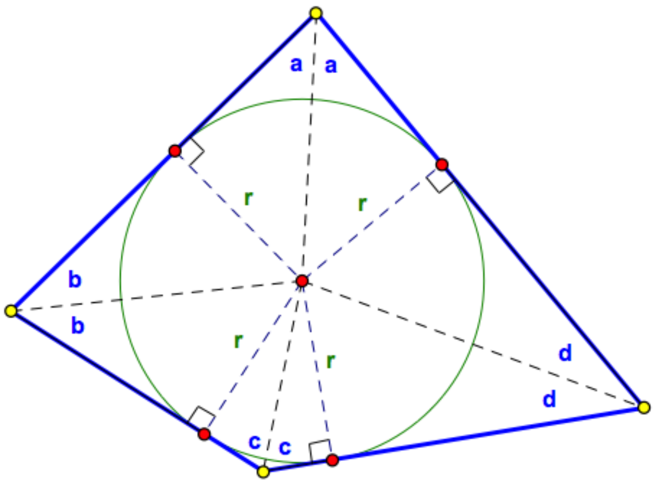

FIGURE 1: Concurrent angle bisectors

Proof:

Since the incentre is equidistant from all four sides (radii of circle are perpendicular to sides), and each angle bisector is the locus of all points equidistant from its two adjacent sides, it follows that each angle bisector must pass through the incentre.

Conversely, one should also note that concurrency of the angle bisectors is a very useful condition for a quadrilateral to be circumscribed around or tangential to a circle. For example, for a quadrilateral to have an incircle, it must have a point that is equidistant from all the sides. Therefore, the four angle bisectors must meet in a single point, i.e. be concurrent.

We actually only need to have three angle bisectors of a quadrilateral concurrent to prove that it is tangential, as the fourth angle bisector will automatically be concurrent with the other three. A teacher may give learners a ready-made geometry sketch to discover and verify this for themselves by dragging the quadrilateral until three angle bisectors are concurrent, and then clicking on a button to view the fourth. An example sketch is available online at: http://dynamicmathematicslearning.com/concurrent-angle-bisectors.html

Theorem 2: If any three angle bisectors of a quadrilateral are concurrent, then the fourth angle bisector is concurrent with them[7] (and hence, the quadrilateral is tangential).

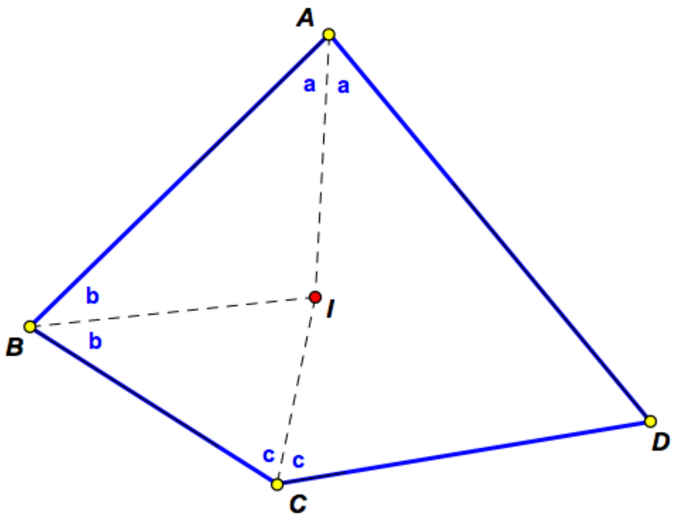

FIGURE 2: More on concurrent angle bisectors

Proof:

With reference to Figure 2, assume the angle bisectors at A, B and C of quadrilateral $ABCD$ are concurrent at I. From the properties of angle bisectors it follows that:

1) I is equidistant from sides AD and AB (lies on angle bisector of $\angle A$)
2) I is equidistant from sides AB and BC (lies on angle bisector of $\angle B$)
3) I is equidistant from sides BC and CD (lies on angle bisector of $\angle C$)

$\Rightarrow$ I is equidistant from AD and CD (transitivity)

$\Rightarrow$ I must lie on the angle bisector of $\angle D$, which completes the proof.

Note from the preceding two theorems it follows that a rhombus (and a square) are tangential since their diagonals are angle bisectors of opposite angles, and hence the incentre of the incircle would be located at the intersection of their diagonals.

Similarly, note that the axis of symmetry of a kite is an angle bisector of a pair of opposite angles. In addition, from the symmetry of the kite, it follows that the angle bisectors of the other two angles (which are reflections of each other) will intersect in a common point on the axis of symmetry. Hence, a kite also has an incircle with its incentre located on the axis of symmetry.

[7] It is easy to generalise Theorems 1 and 2 to any n-gon where having $n-1$ angle bisectors concurrent is sufficient to prove it is tangential. In the case of the triangle, Theorem 2 proves the concurrency of its angle bisectors.

OPPOSITE SIDES

A prescribed theorem in the South African school mathematics curriculum is the following:

A (convex) quadrilateral is cyclic if and only if the opposite angles are supplementary

A different yet equivalent form of formulating this theorem is as follows:

A quadrilateral ABCD is cyclic if and only if $\angle A + \angle C = \angle B + \angle D$

This alternative formulation highlights a remarkable *angle-side* duality between the cyclic quadrilateral and the tangential quadrilateral. Whereas a cyclic quadrilateral has the two sums of opposite *angles* equal, a similar theorem in terms of the two sums of opposite *sides* holds for a tangential quadrilateral.

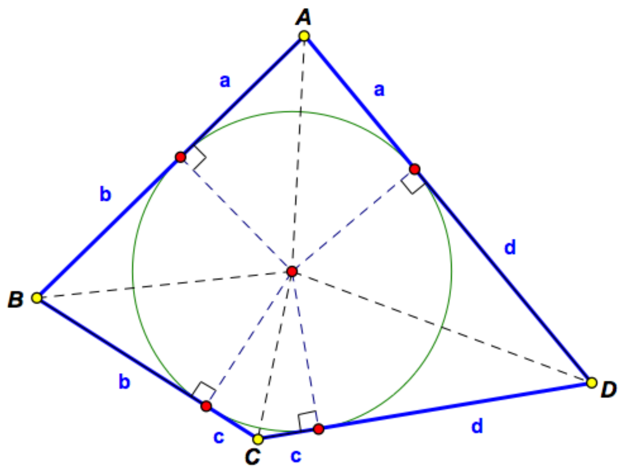

FIGURE 3: Tangential quadrilateral

Pitot's Theorem[8]:

A quadrilateral $ABCD$ is tangential if and only if $AB + CD = BC + DA$

Proof:

The proof of the forward implication is quite straightforward. Assume $ABCD$ is tangential as illustrated in Figure 3. Then labeling equal tangents from the vertices to the circle as indicated it follows that:

$$AB + CD = a + b + c + d = BC + DA$$

A ready-made geometry sketch for learners to dynamically explore and gain confidence in the validity of the converse before dealing with the proof is available at:

http://dynamicmathematicslearning.com/tangent-quad-converse.html

The proof of the converse, like its cyclic counterpart, is probably most easily done via proof by contradiction as follows. Consider Figure 4 where it is given that $AB + CD = BC + DA$. Construct the angle bisectors of angles A and B, and from their point of intersection I drop perpendiculars to sides AB, BC and AD. From I as centre, construct a circle through the feet of these perpendiculars so that sides AB, BC and AD are tangents to it. Assuming CD is not a tangent to this circle, construct a tangent CD' as shown with D' on AD (or its extension).

[8] This theorem is named after a French engineer Henri Pitot (1695-1771) who proved the forward implication in 1725. The converse was proved by the Swiss mathematician Jakob Steiner (1796-1863) in 1846.

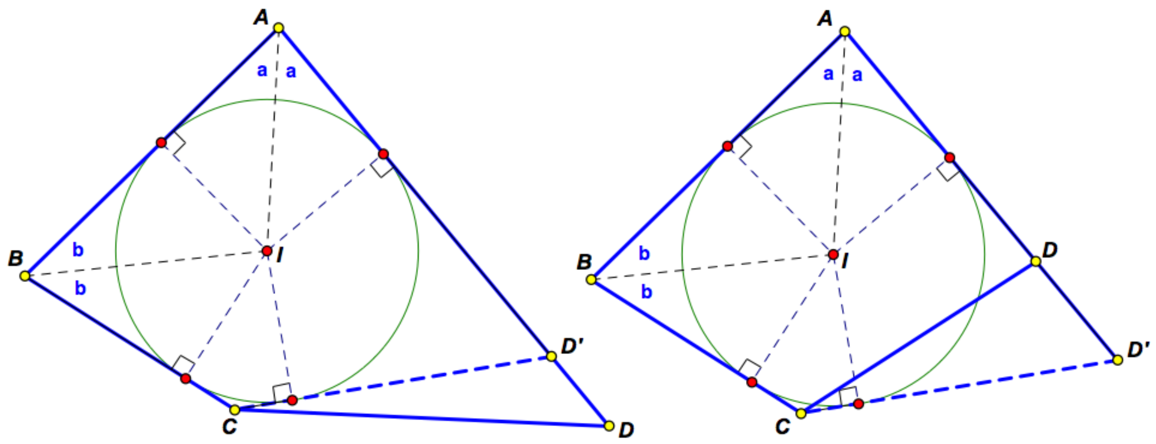

FIGURE 4: Proof of converse of Pitot's theorem

We now have $AB + CD' = BC + AD'$ since $ABCD'$ is tangential. But we are given that $AB + CD = BC + DA$ and therefore $CD' - CD = AD' - AD$ or $CD = CD' - AD' + AD$. However, in the first diagram in Figure 4, $AD = AD' + DD'$ and therefore $CD = CD' - AD' + (AD' + DD') = CD' + DD'$. This is impossible from the triangle inequality unless $DD' = 0$ and D' coincides with D, which contradicts our initial assumption that CD is not a tangent to the circle. It is left to the reader to check that the same contradiction applies to the 2nd case shown in Figure 4. QED.

It should be noted that a tangential quadrilateral could also be concave as shown in Figure 5 (in which case the extension of two of the sides are tangents to the incircle). A dynamic geometry sketch of a tangential quadrilateral, which can be dragged to become concave, is available to explore at:

http://dynamicmathematicslearning.com/tangential-quad.html

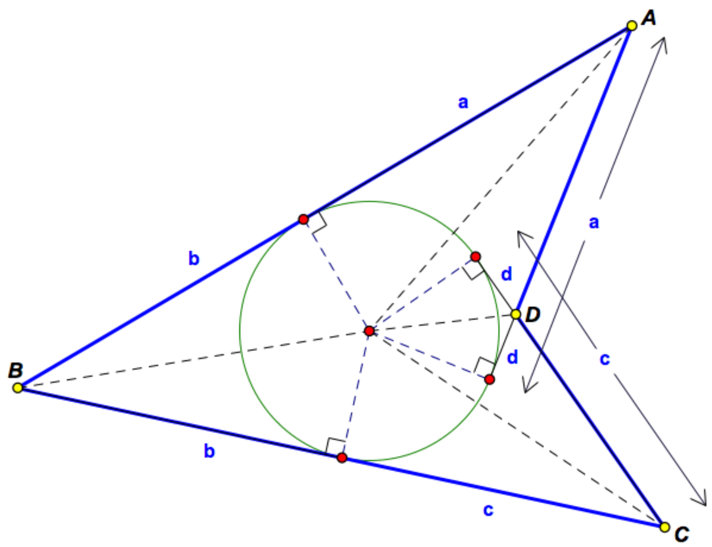

FIGURE 5: Concave tangential quadrilateral

The proof of the concave case shown in Figure 5 is also quite straight forward, since in this case we have $AB + CD = a + b + c - d = BC + DA$. The converse of the concave case can also be proved using proof by contradiction, and is left to the reader.

INCIRCLES

The following result is quite remarkable as it appears to have been discovered only quite recently. A dynamic sketch for learners to first investigate and conjecture the result is available at:

http://dynamicmathematicslearning.com/tangent-incircles-investigate.html

Theorem of Gusić & Mladinić

A quadrilateral is tangential if and only if the incircles of the two triangles formed by a diagonal are tangential to each other[9].

Proof:

If the incircles of triangles ABD and BCD of a quadrilateral $ABCD$ are tangential to one another as shown in Figure 6, then it follows as before that $AB + CD = a + b + c + d = BC + DA$. Therefore, according to Pitot's theorem, $ABCD$ is a tangential quadrilateral. The same argument applies if incircles drawn in the two triangles formed by the other diagonal are tangential to one another.

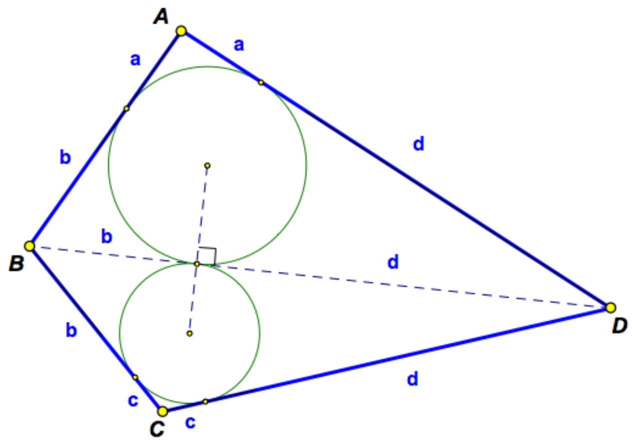

FIGURE 6: Tangential incircles of triangles ABD and BCD

To prove the converse, we shall first prove the following useful general theorem for any convex or concave quadrilateral (and which is also dynamically illustrated in the previous link).

Theorem 3: If a quadrilateral $ABCD$ is divided by a diagonal and the incircles of the two formed triangles are constructed, then the distance k between the two tangent points of the incircles to the diagonal is equal to $|(AB + CD) - (BC + DA)|/2$.

Proof:

Consider Figure 7 which shows a general (convex) quadrilateral with the incircles of triangles ABD and BCD constructed. Labeling the equal tangents to the circles as before, we have that:

$$\frac{|(AB + CD) - (BC + DA)|}{2} = \frac{|(a + b + c + d) - (b + k + c + d + k + a)|}{2} = \frac{2k}{2} = k$$

The concave case is left to the reader.

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[9] This theorem is named after two Croatian colleagues, Jelena Gusić and Petar Mladinić (2001), who as far as I've been able to ascertain, have priority in first publishing the result in 2001 in a journal *Poučak*. Later publications by Worrall (2004) and Josefsson (2011) also mention and prove the theorem.

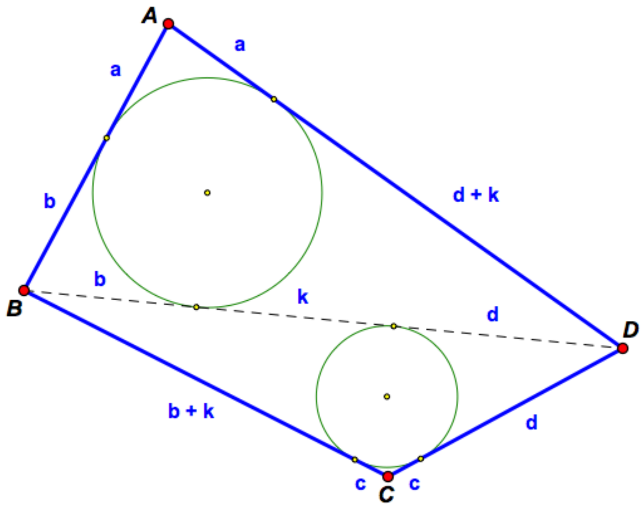

FIGURE 7: Incircles of triangles ABD and BCD

An obvious corollary to Theorem 3 is that the distance between the two tangent points of the incircles of triangles ABC and ACD to the diagonal AC is also equal to k. In addition, from Theorem 3, it now immediately follows that if $ABCD$ is a tangential quadrilateral, then we know from Pitot's Theorem that $AB + CD = BC + DA$, in which case the distance between the two incircles becomes zero, i.e. $k = 0$. In other words, the two incircles are tangential to one another. This then completes the proof of the converse of the Theorem of Gusić & Mladinić.

FURTHER APPLICATION

A neat little application of Theorem 3 is the following result, which can be dynamically explored at:
http://dynamicmathematicslearning.com/tangent-hex-apply.html

Theorem 4: If a tangential hexagon $ABCDEF$ is triangulated by drawing three diagonals from any of its vertices, and the incircles of the four formed triangles are constructed, then the distance between the two tangent points of the incircles to the first diagonal is equal to the distance between the two tangent points of the incircles to the third diagonal.

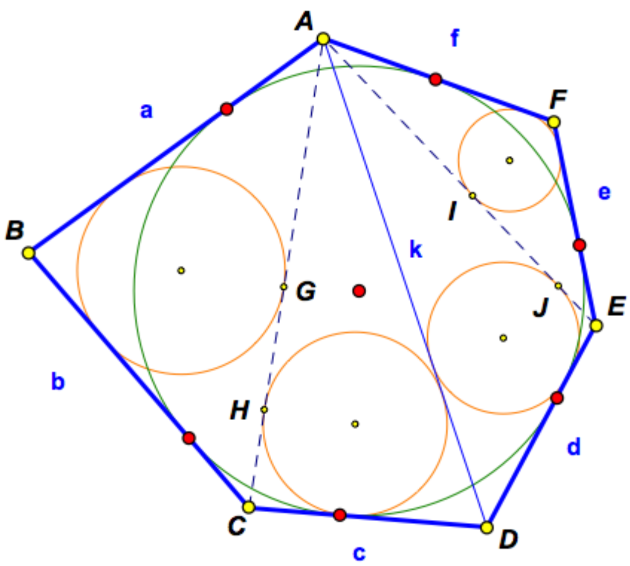

FIGURE 8: A triangulated tangential hexagon with incircles

Proof:

Consider Figure 8 which shows a tangential hexagon $ABCDEF$ with three diagonals drawn from vertex A to triangulate the hexagon. Label the sides of the hexagon consecutively by a, b, c, etc. and the main diagonal AD as k. We shall now use the following generalization of Pitot's theorem proved in De Villiers (1993, 2009), and which follows easily from the equal tangents to a circle theorem: "If a hexagon is tangential, then the two sums of its *alternate* sides are equal."[10]

From the aforementioned theorem, we have $a + c + e = b + d + f$. Next we apply Theorem 3 to $ABCD$ and $DEFA$, and use the alternate side sum result to obtain:

$$2GH = |(a + c) - (b + k)| = |(-e + d + f) - k| = |(d + f) - (e + k)| = 2IJ$$

$$\Rightarrow \quad GH = IJ$$

CONCLUDING REMARKS

This paper has hopefully given the reader some taste of the mathematical possibilities of exploring tangential quadrilaterals. It provides a rich context for revising and applying geometric ideas and theorems such as equidistance, angle bisectors, incircle, incentre, and the equal tangents theorem, as well as the important proof technique of proof by contradiction. Many more beautiful results can be explored and proved regarding tangential quadrilaterals, some of which are given in the references.

Using dynamic geometry software, learners can first experimentally explore several of these results, formulating, checking and disproving conjectures, before engaging in the process of proof. However, some of these results, like the forward implication of Pitot's theorem, could also be meaningfully used to illustrate to learners the *discovery* function of proof, without any prior experimental investigation, by directly applying the equal tangents theorem to a tangential quadrilateral (compare De Villiers, 2003, pp. 68-69).

REFERENCES

De Villiers, M. (1993). A unifying generalization of Turnbull's theorem. *International Journal for Mathematical Education in Science and Technology*, *24*(2), pp. 191-196.

De Villiers, M. (2003). *Rethinking proof with Sketchpad*. Emeryville: Key Curriculum Press.

De Villiers, M. (2009). *Some adventures in Euclidean geometry*. Lulu Publishers.

Gusić, J. & Mladinić, P. (2001). Tangencijalni četverokut. *Poučak*, No. 7, October, pp. 46-53.

Josefsson, M. (2011). More Characterizations of Tangential Quadrilaterals. *Forum Geometricorum*, Vol. 11, pp. 65–82. Accessed on 19 August 2020 at: http://forumgeom.fau.edu/FG2011volume11/FG201108.pdf

Worrall, C. (2004). A Journey with circumscribable quadrilaterals. *Mathematics Teacher*, Vol. 98, No. 3, October, pp. 192-199.

[10] Note that this generalization is true for any tangential $2n$-gon, and what is normally labeled as 'opposite' sides of a tangential quadrilateral can also be regarded as its 'alternate' sides.