Charles Worrall

A Journey with Circumscribable Quadrilaterals

If we want to inspire a talented young writer to be a poet, we do not start with a grammar book. Instead, inspiration and love of literature come from reading complex works of art like *Hamlet* or *Catcher in the Rye*. Promoting a similar idea, Einstein wrote, "the mysterious . . . is the source of true art and science" (Ulam 1976, p. 289). To be more specific, one way to develop urgency and curiosity in mathematics students is to encourage them to wrestle with complex mathematical objects, ones whose deep and mysterious relationships are waiting to be found.

Recently, my honors geometry students and I investigated a mysterious object that rewards close scrutiny and fosters curiosity in students. The object first came to my attention years ago as a dead-end problem meant to test knowledge of the following simple circle theorem:

(1) *The two tangent segments drawn from a point outside a circle to the circle are congruent.*

The problem illustrated in **figure 1** is probably familiar to geometry teachers:

> PROBLEM 1*:* A quadrilateral with consecutive sides of length 12, 15, and 17 units circumscribes a circle. Find the length of the fourth side.

I use the term *dead-end* to mean that the problem was probably designed only to illustrate the algebraic implications of theorem (1), not to spur any further questioning. A little pushing and prodding, though, reveal a doorway to some fascinating mathematics. In the remainder of this article, I trace a path beginning with problem 1 that illustrates how teachers can use their own mathematical curiosity to engender the same in students, thereby showing where a simple but relentless habit of questioning can lead.

A minor trick makes solving problem 1 simple. If we resist the urge to assign a variable to AD and instead let one of the small tangent segment lengths be x, then by using theorem (1) and segment addition repeatedly, we can rewrite all the side lengths in terms of x. **Figure 2** shows a particular case. We see that

$$\begin{aligned} AD &= (14 - x) + x \\ &= 14. \end{aligned}$$

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When students first solve a problem like this one by seeing the x's drop out, they almost jump out of their seats. We might as well pull a rabbit out of a hat, because it seems like magic, not mathematics.

But more interesting than the magical algebraic properties of the solution is the set of questions—the *pathway*—that the solution opens up for students. If prompted to think beyond finding the "solution" to bigger questions of causality, at least one keen student will usually notice that $14 + 15 = 12 + 17$; and he or she will perhaps even wonder whether the sum of a pair of opposite side lengths is always equal to the sum of the other pair. A polygon (apparently) with a property as surprising as this one is worthy of a definition. So we define a *circumscribable quadrilateral* to be one that contains a circle tangent to each of its sides.

Questions arise. Are all quadrilaterals circumscribable? If not, then what characterizes one that is and one that is not? Can we prove generally the constant sum of opposite sides observed in the problem? What does *generally* mean here? Does an extension to circumscribable n-gons exist? All these questions really boil down to one overarching one, which I phrase in the following two ways:

- What are the properties of circumscribable quadrilaterals; and conversely, what are the necessary conditions that make quadrilaterals circumscribable?
- Or, as I state the question in class, What makes these things tick?

Different mathematicians will proceed in different ways, and other fruitful extensions and alternative avenues of inquiry can be found. But I find that beginning by looking for insights in a simpler class of objects than quadrilaterals, namely, circumscribable triangles, is useful here. Of course, the famous *incircle* of a triangle and its center, the *incenter,* have their own set of marvelous properties, all of which begin with the fact that in any triangle, the incenter sits at the point of concurrency of the three angle bisectors. Ironically, though, as an answer to the question that we are investigating, the fact that all triangles have an incircle is actually a bit disappointing. What is the defining characteristic of a circumscribable triangle? Apparently, simple existence is the answer.

We can learn some lessons from the triangle case. First, we should think about the angle bisectors in a circumscribable quadrilateral. Second, with more urgency than before, we must ask—and hope the answer is no—whether all quadrilaterals are circumscribable. If we allow nonconvex quadrilaterals, that is, quadrilaterals with angles of more than 180 degrees, the answer is definitely no. A little thought reveals that no

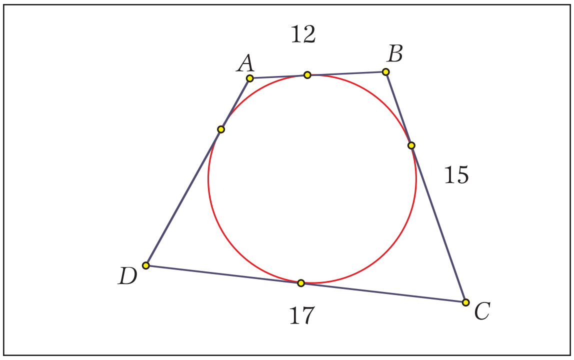

Fig. 1 Problem 1: Find *AD*.

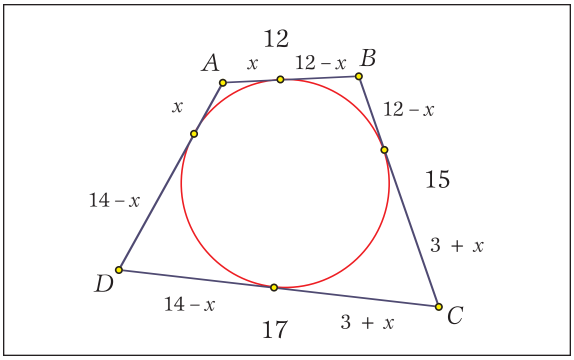

Fig. 2 Side lengths rewritten in terms of *x*

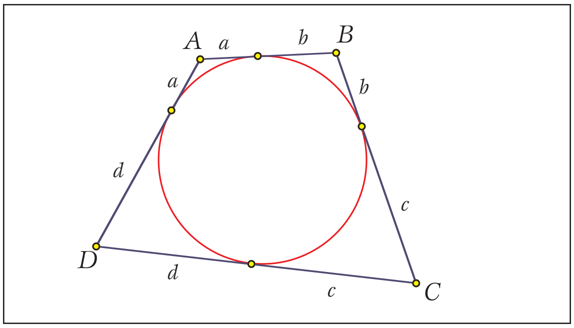

Fig. 3 A circumscribable quadrilateral

nonconvex quadrilateral can possibly be circumscribable. So the real question is whether all convex quadrilaterals are circumscribable. In this article, we assume everywhere that *quadrilaterals are convex.*

To deal with the second of these lessons first, a moment's reflection convinces us that, at the very least, a square is circumscribable. It is fair to say, therefore, that at least a few such quadrilaterals exist. Consider in **figure 3** a circumscribable quadrilateral $ABCD$ (not necessarily a square, of course). Each side of the quadrilateral is divided naturally into two parts by a point of tangency to the circle. By theorem (1), the side lengths $a, b, c,$ and d occur in pairs as they appear in the diagram. We easily see that

$$\begin{aligned} AD + BC &= (a + d) + (b + c) \\ &= (a + b) + (d + c) \\ &= AB + CD. \end{aligned}$$

Phrased nicely, the theorem says,

(2) *In a circumscribable quadrilateral, the sum of the lengths of a pair of opposite sides is equal to the sum of the lengths of the other pair of opposite sides.*

Hooray! Circumscribable quadrilaterals do in fact have the equal-sum property. Just as exciting is that we have proved definitively that not all quadrilaterals are circumscribable. Theorem (2) is originally from Darij Grinberg, a brilliant German university student. His proofs are available at de.geocities.com/darij_grinberg. The circumscribable quadrilaterals are in an elite club, every member of which has the special property described. But does every quadrilateral with the equal-sum property get to be in the club? We would love to prove the following theorem, since it seems like it ought to be true.

(3) *If the sum of the lengths of a pair of opposite sides of a quadrilateral is equal to the sum of the lengths of the other pair of sides, then the quadrilateral is circumscribable.*

At least three completely different proofs of the problem exist. One is by contradiction, one uses the Pythagorean theorem, and a third—the most direct and beautiful—is found in a dusty old mathematics textbook. The proof by contradiction is my own. The proof that uses the Pythagorean theorem is by Christopher Jones at the Horace Mann School in New York City. The dusty old mathematics textbook is *College Geometry,* by Altshiller-Court (1952). A particular class might find its way to one of these proofs or another one altogether. In my class, I sometimes take the time with a theorem like this one to assign a two-day research project. Students team up in small groups, each studying one of the several proofs, and then they give fifteen-minute class presentations.

In his classic *College Geometry,* Nathan Altshiller-Court (1952, p. 135–36) proves theorem (3) in a way that makes apparent deep relationships inside circumscribable quadrilaterals; it is therefore the one that we will consider. I present the proof in my own words, beginning with two common theorems.

(4) *The three perpendicular bisectors of the sides of a triangle are concurrent at a point called the* circumcenter *of the triangle.*

(5) *A point is on the angle bisector of an angle if and only if the point is equidistant from the sides of the angle, that is, if and only if the lengths of the perpendicular segments from the point to the sides of the angle are equal.* (See **fig. 4.**)

We forgo the proofs of theorems (4) and (5) since they are found in many other places.

Before beginning the proof of theorem (3), we can use theorem (5) to say something important about the angle bisectors in a circumscribable quadrilateral—as we resolved to think about a moment ago when we were considering triangles. (See **fig. 5.**) Suppose that in a circumscribable quadrilateral *ABCD,* we draw the inscribed circle, its center *O,* and the four radii to the sides. The radii are clearly congruent to one another and perpendicular to the sides of *ABCD*. From theorem (5), we immediately conclude that *O* is on each of the angle bisectors of the angles of *ABCD.* Conversely, suppose that a quadrilateral *ABCD* has four angle bisectors concurrent at a point *O*. Then by theorem (5) again, the four perpendicular segments from *O* to the sides of *ABCD* are of equal length. If we draw in the circle with center *O* through the feet of the four perpendicular segments, then, by their perpendicularity to the segments, the sides of *ABCD* are tangent to the circle. Thus, *ABCD* circumscribes circle *O,* making *ABCD* a circumscribable quadrilateral. We have proved a theorem reminiscent of those for the incircle and incenter of a triangle.

(6) *The four angle bisectors of a quadrilateral are concurrent if and only if the quadrilateral is circumscribable.*

We are ready for Altshiller-Court's proof of theorem (3). It is divided into two cases that at first glance do not seem particularly helpful: the case when two consecutive sides of *ABCD* are of equal length and the case when they are of different lengths. We begin with *ABCD* such that $AB + CD = AD + BC,$ and in case 1, where $AB = BC$. We must show that *ABCD* is circumscribable. With that goal

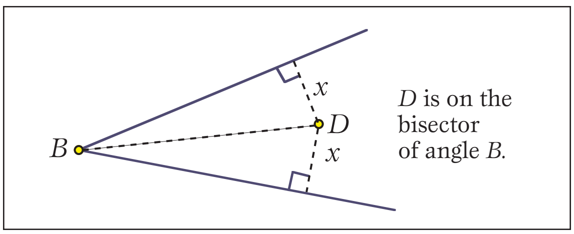

Fig. 4 Illustration of theorem (5)

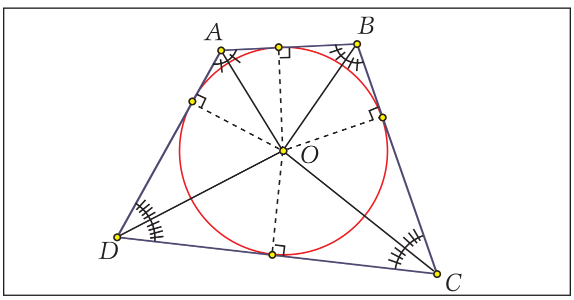

Fig. 5 Angle bisectors in a circumscribable quadrilateral

in mind, we subtract the second equation from the first to see that $CD = AD$, too. Therefore, as **figure 6** indicates, $ABCD$ must be either a kite or a rhombus. Either way, after drawing $\overline{BD}$, we can see that $\triangle ABD \cong \triangle CBD$ by SSS, and so $\angle B$ and $\angle D$ are each bisected by $\overline{BD}$. When we consider the same congruent triangles, the angle bisectors of the corresponding angles at vertices A and C must intersect the common opposite side, $\overline{BD}$, at the same point, labeled O in the figure. We have therefore shown that the four angle bisectors are concurrent at O; and so, by theorem (6), $ABCD$ is circumscribable.

Although case 1 is relatively commonplace (after all, congruent triangles are the bread and butter of high school geometry), the proof of case 2 is downright spectacular. We are again given that $AB + CD = AD + BC$, but we assume that $AB \neq BC$ and without loss of generality, that $AB < BC$. By inspecting the equation $AB + CD = AD + BC$, we conclude that $CD > AD$. Therefore, there exist points E and F on $\overline{BC}$ and $\overline{CD}$ such that $BE = AB$ and $FD = AD$. From the given information, segment addition, substitution, and subtraction, we obtain the following equations:

$$\begin{aligned} AB + CD &= AD + BC \\ \Rightarrow AB + CF + FD &= AD + BE + EC \\ \Rightarrow BE + CF + AD &= AD + BE + EC \\ \Rightarrow CF &= EC \end{aligned}$$

Inspecting **figure 7** indicates that, perhaps surprisingly, $\triangle ABE$, $\triangle ADF$, and $\triangle ECF$ are each isosceles. Right here, we might pause to acknowledge that we have found a part of the "nature" of circumscribable quadrilaterals—a part of what makes them tick. They all contain three isosceles triangles.

We next consider the three respective angle bisectors at $\angle B$, $\angle C$, and $\angle D$. Since $\triangle ABE$ is isosceles, the angle bisector at $\angle B$ is also the perpendicular bisector of base $\overline{AE}$. By the same reasoning, since the angle bisectors at $\angle C$ and $\angle D$ are in isosceles triangles, too, they are the perpendicular bisectors of bases $\overline{EF}$ and $\overline{AF}$. Thus, the three angle bisectors at $\angle B$, $\angle C$, and $\angle D$ are simultaneously the three perpendicular bisectors of the sides of $\triangle AEF$, which by theorem (4) are concurrent at a point O, the circumcenter of the triangle. We have three of the angle bisectors of $ABCD$ concurrent. Now we hope that the fourth angle bisector goes through O, too. This part is easy: in **figure 8,** we draw in the four perpendicular segments from O to the sides of $ABCD$. (A fine discussion develops if a student asks how we can be sure that the perpendiculars fall inside $ABCD$.) By using theorem (5) three times and some transitivity, we find that O is equidistant from sides $\overline{AD}$ and $\overline{AB}$. Therefore, again by theorem (5), O is on the bisector of $\angle A$, which makes $ABCD$ circumscribable, by theorem (6). QED.

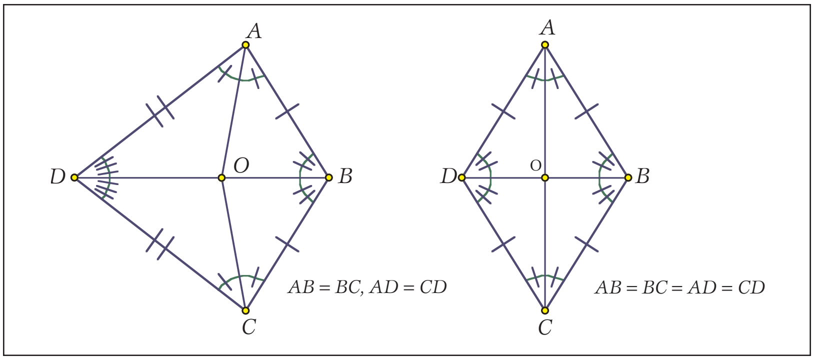

Fig. 6 Each of the angle bisectors of *ABCD* is concurrent at point *O*.

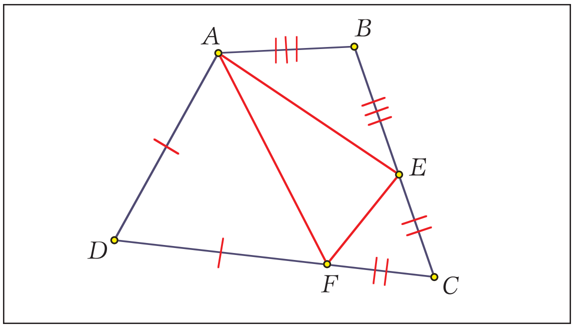

Fig. 7 Triangles *ABE*, *ADF*, and *ECF* are each isosceles.

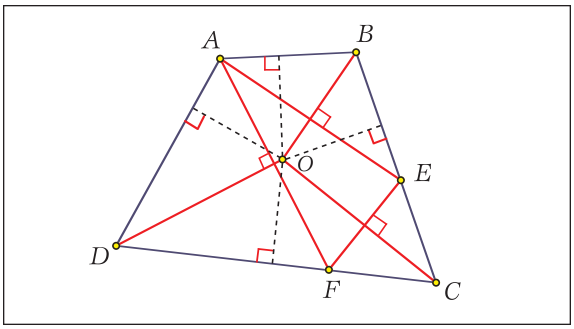

Fig. 8 Draw four perpendicular segments from point *O* to the sides of *ABCD*.

The style of most geometry textbooks lulls us into a tendency to make a full stop after finding such a fine result; but typically in mathematics, this moment is exactly when the fun begins. In fact, the isosceles triangles in the Altshiller-Court proof are extremely provocative, strongly hinting that more structure waits to be observed in circumscribable quadrilaterals. I have found at least one more necessary and sufficient condition for circumscribability. The theorem does not exist in any of the literature that I have read on circumscribable quadrilaterals, but it instead arose from a problem that I designed to give students the chance to use theorem (1). It is another good example of an ostensible ending point that can serve as a beginning (the first example being problem 1).

PROBLEM 2: In **figure 9,** the incircles of $\triangle ADC$ and $\triangle ABC$ are tangent to each other at E. If $AD = 15$, $DC = 17$, and $BC = 16$, then find AB.

Just as with problem 1, the solution is not difficult if we assign a variable to a tangent length and follow our noses. We let $AE = x$, for example. Then the point of tangency with the incircle divides $\overline{AD}$ into two lengths, x and $15 - x$. Continuing with the same technique, we find expressions for the lengths on $\overline{CD}$, then $\overline{CE}$, then $\overline{BC}$, and finally $\overline{AB}$, as indicated in **figure 10.** The length of $\overline{AB}$ is $x + (14 - x)$, which is 14.

But there is something fishy about that answer—not to mention familiar about the method of solution. Given a moment to think, someone will notice that

$$15 + 16 = 17 + 14.$$

The sum of one pair of opposite side lengths is equal to the sum of the other pair of opposite side lengths. Although its inscribed circle is nowhere to be seen, it is a circumscribable quadrilateral. Surprisingly, what we have is another necessary and sufficient condition for circumscribability. Specifically,

(7) *A quadrilateral is circumscribable if and only if the incircles of the two triangles formed by a diagonal are tangent to each other.*

In **figure 11,** we start the first half of the proof given quadrilateral $ABCD$, diagonal $\overline{AC}$, and the incircles of the resulting triangles tangent at point E. We must show that $ABCD$ is circumscribable. The lengths $a, b, c,$ and d appear in the diagram indicating equal lengths. It should be clear, though, that the three segments labeled a are in fact equal because two pairs of segments are congruent by theorem (1). All three are congruent only because of transitivity, because they are each congruent to the shared tangent $\overline{AE}$ of length a. The same argument provides justification for concluding that the three sides of label c in the figure are in fact congruent.

The proof is now immediate:

$$\begin{aligned} AB + CD &= (a + b) + (c + d) \\ &= (a + d) + (b + c) \\ &= AD + BC \end{aligned}$$

By theorem (3), $ABCD$ is circumscribable.

Recalling how clever Altshiller-Court needed to be to prove the second half of his theorem, we might be hesitant to try the second half of the proof of theorem (7). But it turns out to be remarkably clean and simple. We are given a circumscribable quadrilateral $ABCD$ and must show that the incircles of the triangles formed by diagonal $\overline{AC}$ are tangent to each other.

We consider the two incircles of $\triangle ACD$ and $\triangle ACB$ in **figure 12,** and we do not picture the circles' points of tangency on $\overline{AC}$. Furthermore, from theorem (1), we have only two congruences, so we have the lengths $a, b, c, d, e,$ and f, as they are marked in the figure.

Since $ABCD$ is circumscribable,

$$AB + CD = AD + BC.$$

So

$$(a + b) + (e + d) = (e + f) + (b + c).$$

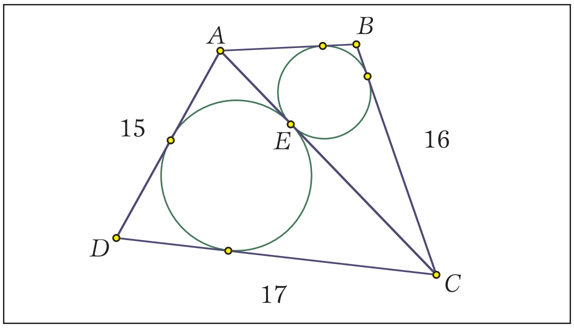

Fig. 9 Problem 2: Find *AB*.

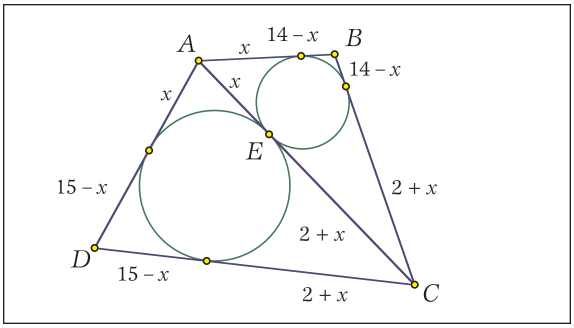

Fig. 10 Segment lengths rewritten in terms of *x*

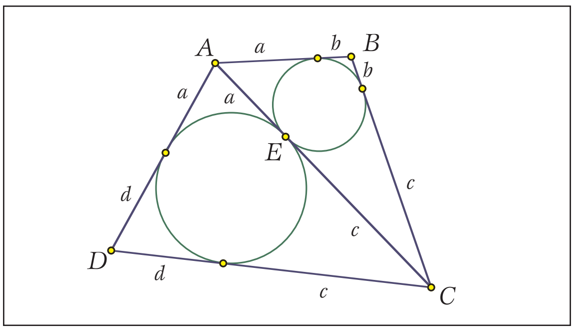

Fig. 11 The incircles of *ABCD* are tangent to each other. Show that *ABCD* is circumscribable.

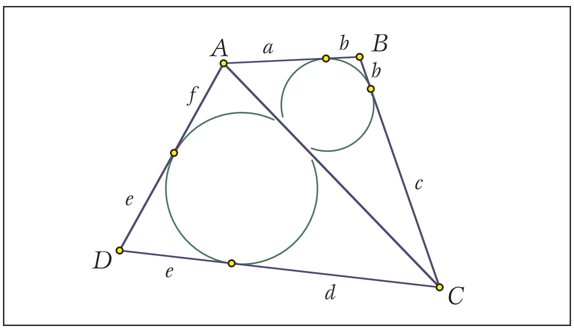

Fig. 12 Quadrilateral *ABCD* is circumscribable. Show that the incircles are tangent to each other.

By subtracting b and e from both sides, we get

(*) $$(a+d)=(c+f).$$

Since $\overline{AC}$ is tangent to the incircle of $\triangle ABC$, we find that $AC=(a+c)$ by applying theorem (1) twice. By the same reasoning, since $\overline{AC}$ is tangent to the incircle of $\triangle ACD$, $AC=(d+f)$. So

(**) $$(a+c)=(d+f).$$

By subtracting equation (*) from equation (**), we get

$$(a+c)-(a+d)=(d+f)-(c+f).$$

Therefore,

$$c-d=d-c$$

and

$$2c=2d.$$

Finally,

$$c=d.$$

Looking back at **figure 12,** we note that $c=d$ means that the distances from C to the points of tangency of both the incircles on side $\overline{AC}$ are equal. Therefore, the points are the same, and our proof of theorem (7) is complete. QED.

To sum up, we have found three different necessary and sufficient conditions for quadrilateral circumscribability. Said another way, if one of the following four statements is true, then all of them are true:

- $ABCD$ is circumscribable.
- The four angle bisectors of $ABCD$ are concurrent.
- $AB+CD=AD+BC$.
- The incircles of the two triangles formed by a diagonal of $ABCD$ are tangent to each other.

Figure 13 summarizes the discoveries that we have made. When presented to a class or constructed by students themselves by using such interactive geometry software as The Geometer's Sketchpad, the diagram makes a visually satisfying answer to the overarching questions: What are the properties of a circumscribable quadrilateral, and what are the necessary conditions for its existence? Seeing the five circles generated, we bear witness to what makes circumscribable quadrilaterals tick.

A neat ending of this journey is again possible, but—again—why stop? To show some more of the richness of these quadrilaterals—and of the implications of the Altshiller-Court proof—I will prove one more theorem. In a circumscribable quadrilateral, there are two other interesting circles, both of them concentric with the quadrilateral's inscribed circle. I actually found seven interesting concentric circles centered at O, which, for those inclined to further investigation, can easily be found by fiddling with The Geometer's Sketchpad. The theorem is a little hard to state, but it can be simply understood with a picture.

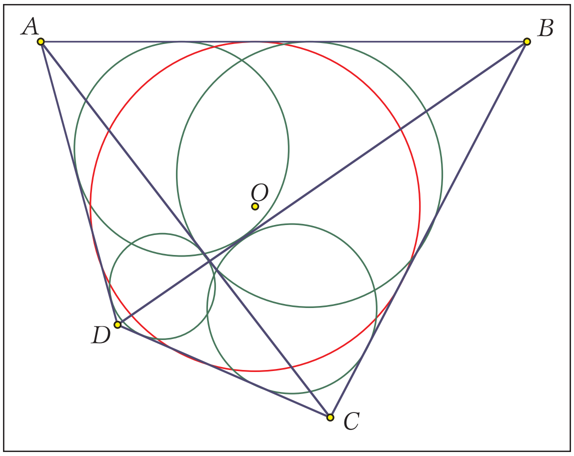

Fig. 13 Summary of discoveries about quadrilateral circumscribability

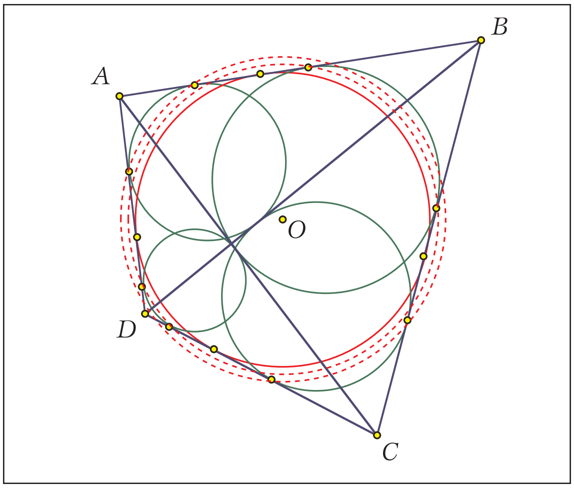

Fig. 14 Two circles that are concentric with the quadrilateral's inscribed circle

(8) *The four sides of a circumscribable quadrilateral intersect the incircles of the two triangles formed by a diagonal of the quadrilateral to form four points, all of which are on a circle that is concentric with the quadrilateral's inscribed circle.*

Of course, since there are two diagonals, there are two such circles, both concentric with the original inscribed circle, as shown in **figure 14.**

The proof is simple and satisfying. Perhaps not unexpectedly, we will make use of the circumscribable quadrilateral's more famous cousin, the cyclic quadrilateral, one for which a circle exists that passes through all four vertices. At this point, we need just two things: a simple algebraic observation about the lengths of segments created by the points of tangency of the five circles inside a circumscribable quadrilateral, and a well-known corollary to the inscribed-angle theorem for circles. First, the corollary:

(9) *Two opposite angles in a quadrilateral are supplementary if and only if the quadrilateral is cyclic.*

The proof is available in many other places, so we omit it here.

The observation about side lengths is actually a repeated application of theorem (1). It is a nice exercise for students to convince themselves that each side of circumscribable quadrilateral *ABCD* is divided into four of six lengths marked *a, b, c, d, e,* and *f,* as in **figure 15.** Take a moment to convince yourself, as well.

We next consider $\triangle DAB$. With the memory of Altshiller-Court's proof still vivid, it is almost impossible not to see in **figure 15** that three nested isosceles triangles are in $\triangle DAB$, their bases forming three parallel lines. **Figure 16** shows those and three other sets of three parallel lines, corresponding to similar nestings in $\triangle BCD$, $\triangle CDA$, and $\triangle CBA$. (Once again, I am tempted to say that this one, **figure 16,** shows the essential qualities of circumscribable quadrilaterals.)

The proof is almost anticlimactic from here. Making the trivial observation that *FGHJ* is cyclic, we see that $\angle JFG$ and $\angle JHG$ are supplementary by corollary (9). From the parallel lines, we easily see that $\angle JFG \cong \angle SKN$ and that $\angle JHG \cong \angle SPN$. Therefore, $\angle SKN$ is supplementary to $\angle SPN$, and *SKNP* is cyclic by corollary (9), as shown in **figure 17.** By the way, if we look around, we can find a lot of other cyclic quadrilaterals—most, though, are not centered at *O;* readers who are of a combinatorical bent might find that counting them is an interesting challenge. All that remains is to show that *O* is the center of the circle that circumscribes *SKNP.*

The perpendicular bisector of a chord of a circle always passes through the circle's center. Therefore, since *O* is known to be the center of the circle that circumscribes *FGHJ,* we conclude that *O* is the point of intersection of the perpendicular bisectors of $\overline{FJ}$ and $\overline{JH}$. To show that *O* is the center of the circle through *SKNP,* showing that *O* is the point of intersection of the perpendicular bisectors of $\overline{SK}$ and $\overline{PS}$ is sufficient. As we know, the bisector of the vertex angle of an isosceles triangle is also the perpendicular bisector of the opposite side. Therefore, since $\overline{AO}$ is the perpendicular bisector of $\overline{FJ}$, it is also the angle bisector of $\angle A$, which makes it the perpendicular bisector of $\overline{SK}$ in isosceles $\triangle SAK$. By identical reasoning, $\overline{DO}$, the perpendicular bisector of $\overline{JH}$, is also the perpendicular bisector of $\overline{PS}$. Therefore, we have shown that the perpendicular bisectors of $\overline{SK}$ and $\overline{PS}$ intersect at *O,* making it the center of the circle circumscribing *SKNP*. QED.

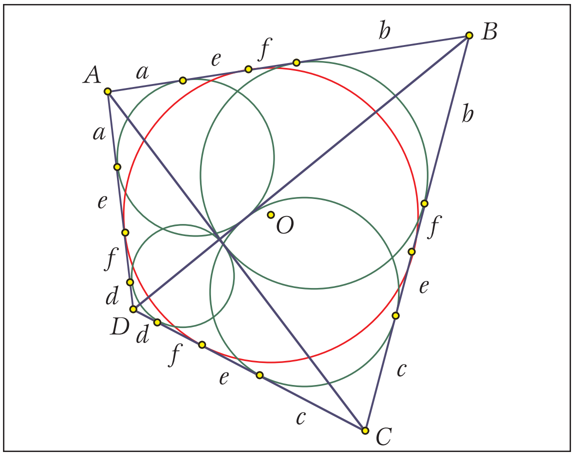

Fig. 15 Each side of circumscribable quadrilateral *ABCD* is divided into four of six lengths marked *a, b, c, d, e,* and *f.*

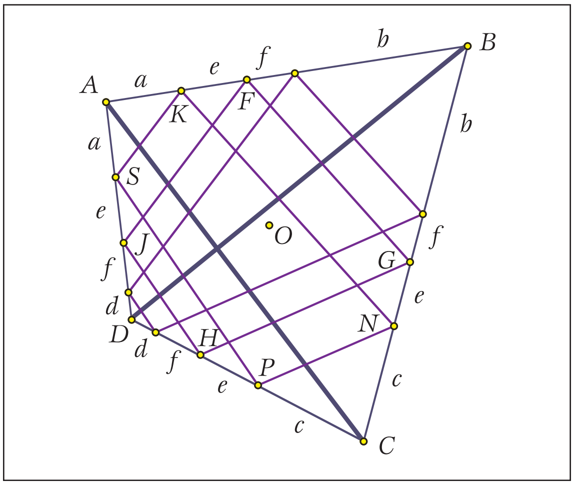

Fig. 16 Three nested triangles are in triangles *DAB, BCD, CDA,* and *CBA,* their bases each forming three parallel lines.

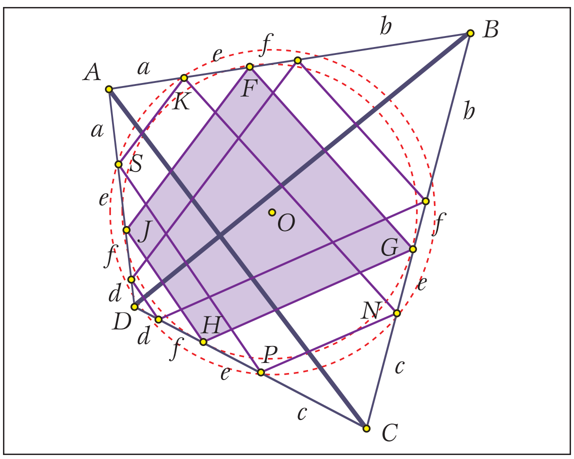

Fig. 17 Quadrilateral *SKNP* is cyclic.

We have not exhausted the possibilities for theorems in circumscribable quadrilaterals—which is perhaps my overall point in writing this article. One of the keys to teaching our students to be curious and deep thinkers is to illustrate to them an easily misunderstood fact: the reason that high school teachers move from one topic to the next is that we must, because we teach survey courses. In fact, we never move to the next topic because we have run out of interesting things to investigate. In this light, we are tempted to end each unit with an apology. "I am sincerely sorry that we must now stop investigating circumscribable quadrilaterals. We could spend years finding surprising, deep results here, but we do not have the time in this course. Thus, we reluctantly move on. However, if any of you finds something else interesting in your free time, please tell me about your discoveries. Perhaps we could coauthor a paper." The payoff to grappling with "mysterious" objects like circumscribable quadrilaterals might not only be more curious students, but published ones, as well.

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EDITORS' NOTE

When we read submissions to "Delving Deeper," we are always on the lookout for interesting directions in which readers can take the author's ideas in yet other directions. We found ourselves in a nice e-mail dialogue with the author of this article, part of which we share here.

Eds: Dear Charles,

There are several places where your article tempts one into excursions into projective geometry. For example, when you shift attention from segments tangent to a circle to points on the circle, you play with a duality that might be interesting for some readers to follow up. For that matter, imagine your construction drawn on a piece of paper lying on a desk and viewed from an angle (at sufficient distance so that you can treat the projection as taking rectangles to parallelograms instead of the trapezoids that "real" local perspective generates). The circle becomes an ellipse, around which you have circumscribed a quadrilateral. All the things that you have proved have a generalization in this world of more-general conics.

Author's reply: There certainly is a generalization to conics—parabolas and hyperbolas, as well as ellipses. I've really fallen in love with these little quadrilaterals because there are so many directions to go with them. My own follow-up has remained squarely in traditional Euclidean geometry, where a boatload of observations can still be made. I am a member of an e-mail forum called *Hyacinthos* (it's a Yahoo group that anyone can join), which is an extraordinary clearinghouse for all things Euclidean, especially the advanced geometry of the triangle. In December, I responded to a mathematician's ideas about a circumscribable q-lateral that he was looking at, and it ended up generating a remarkable flurry of e-mails from around the world. (It also corroborated that my final theorem in the article is probably new.) Lots of amazing theorems can be proved about these little objects. Here are two especially noteworthy theorems:

1. *If ABCD is a circumscribable q-lateral, then the incenters of the four nonoverlapping triangles formed by ABCD and the intersection of its diagonals are concyclic.*

It is a marvelous theorem for two reasons: it is simple to state and see, but it turns out to be difficult to prove; and in looking for a proof, we all turned up a zillion other surprising facts about the center of that circle, tangents to it, strange relationships between parallel lines, and so on. It is a doorway into some amazing stuff.

The second theorem is much less important—or fertile—but the proof that I saw for it uses some advanced ideas that left me, at least, feeling pretty unsatisfied. Then I started to play a little on my own and found a proof that is based on the theorem in the article about the common tangent point for the incircles of ABC and ADC in circumscribable q-lateral $ABCD$. Here is the theorem:

2. *If ABCD is a circumscribed quadrilateral with incenter O, the perpendicular to AB through A meets BO at M, and the perpendicular to AD through A meets DO at N, then MN is perpendicular to AC.*

My proof: [The author supplied a particularly elegant proof, which the editors have mischievously omitted. Pick a direction—Euclidean or projective—and have fun! —Eds.] ∞

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