Thabit's Generalisation of the Theorem of Pythagoras

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INTRODUCTION

There are many possible generalizations of the theorem of Pythagoras. Some of the most well known ones are the cosine formula, the distance in n dimensions, and Ptolemy's theorem. These three generalisations, along with seven others, are discussed in De Villiers (2009, pp. 69-75). One of these other generalizations deserves to be better known and was apparently first proven by the Turkish scientist Thabit Ibn Qurra in approximately 900 AD.

THABIT'S GENERALISATION

Let ABC be any triangle. Construct lines AD and AE as shown in Figure 1 so that angles BDA and CEA are both equal to angle BAC. Then $AB^2 + AC^2 = BC(BD + CE)$. The reader is invited to dynamically explore the result at http://dynamicmathematicslearning.com/thabit-pyth-generalization.html.

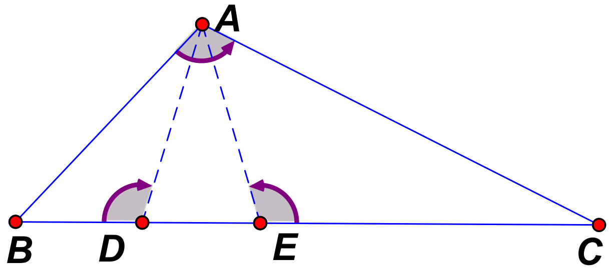

FIGURE 1

PROOF

From the construction it follows that triangle ABC is similar to triangle DBA as well as EAC. From the similarity of triangles ABC and DBA we have $\frac{AB}{BD} = \frac{BC}{AB} \Rightarrow AB^2 = BD \cdot BC$. From the similarity between triangles ABC and EAC we have $\frac{AC}{EC} = \frac{BC}{AC} \Rightarrow AC^2 = EC \cdot BC$. Adding these two equations gives us the required result: $AB^2 + AC^2 = BC(BD + EC)$.

A FEW BRIEF COMMENTS

1) An easy way of constructing angles BDA and CEA equal to angle BAC is simply to rotate lines AB and AC around A respectively by the *directed* angles ACB and ABC.

2) When the angle at vertex A is a right angle, angles BDA and CEA will also be right angles, and the points D and E will coincide (since the perpendicular from A to BC is unique). In other words, we then obtain the theorem of Pythagoras, namely $AB^2 + AC^2 = BC^2$.

3) The proof of Thabit's generalization is identical to one of the standard ways of proving the theorem of Pythagoras using similar triangles, and provides a very accessible generalisation with which to challenge learners.

4) To improve their visualisation and critical reasoning skills, learners ought also to be challenged to apply/check the proof of Thabit's generalization in different configurations, as illustrated in Figure 2.

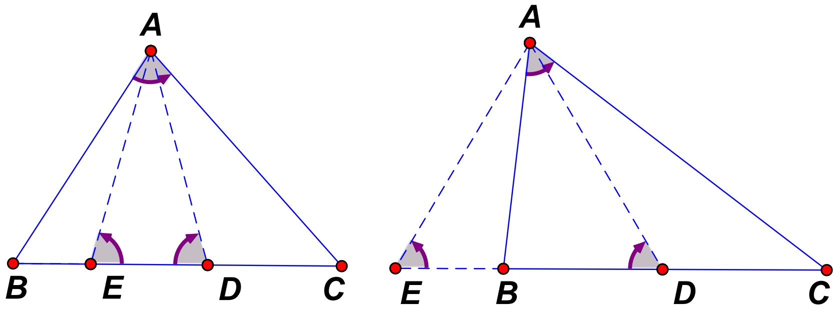

FIGURE 2

Though not the case here, proofs sometimes need to be adapted for other configurations, for example the three different configurations that need to be considered in proving the theorem that states that the angle subtended at the centre of a circle is twice the subtended angle on the circumference. It is good mathematical practice always to check other possible configurations.

5) The generalization can also be nicely displayed as shown in Figure 3, where the sum of the areas of the squares on sides *AB* and *AC* are equal to the area of the rectangle constructed on *BC*, with the other side equal to *BD* + *EC*.

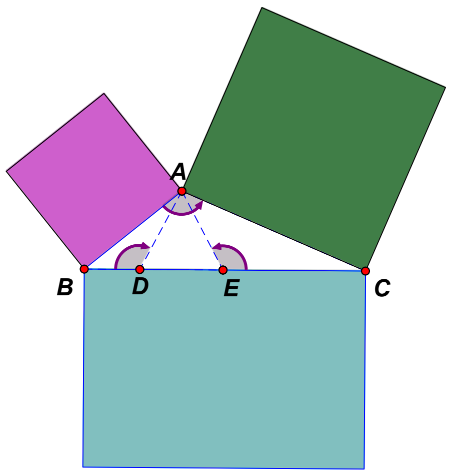

FIGURE 3

6) With a dynamic sketch, or some logical thinking, learners could also easily explore/determine the respective conditions under which *BC* is greater than or smaller than *BD* + *EC*.

REFERENCES

De Villiers, M. (2009). *Some Adventures in Euclidean Geometry*. Lulu Publishers.