A Theorem on Tangent Cycles

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During the Edo period (1603–1867) the Tokugawa Shogunate closed Japan to foreigners. In this era, Japanese mathematics thereby developed practically independently of Western science. We call this mathematics *wasan*. Wasan geometry left us many results on tangent circles. Let us take up two results from among them: In Figure 1 $r_1r_3 = r_2r_4$ (Aida [1]), and in Figure 2 $r_1r_3r_5r_7 = r_2r_4r_6r_8$ (Baba [2]), where r_i is the radius of the circle C_i.

These results have been explained severally up to the present. In this note we will show that each of the two results is a special case of one generalised theorem. For this purpose, we add orientations to circles and lines described by arrows. An oriented circle and a line are called a cycle and a ray. They are said to be tangent to each other if they touch as a circle and a line, and the orientations at the point of tangency are the same. Two rays are parallel if they are parallel as two lines and have the same orientations. For three rays x, y, z, if each pair of two rays are not parallel, they have a unique common tangent cycle. We

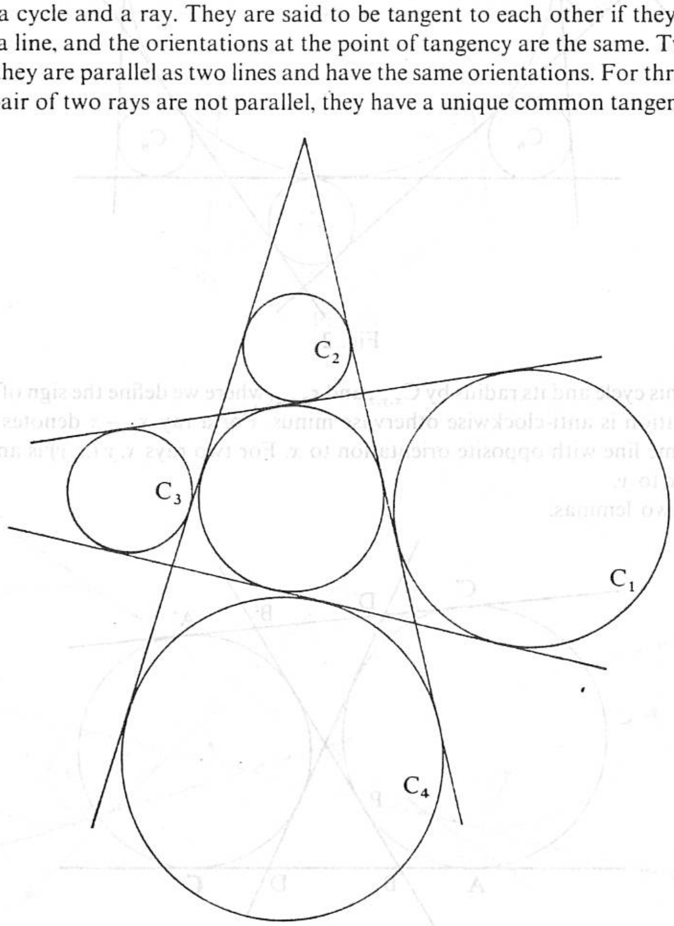

Fig. 1

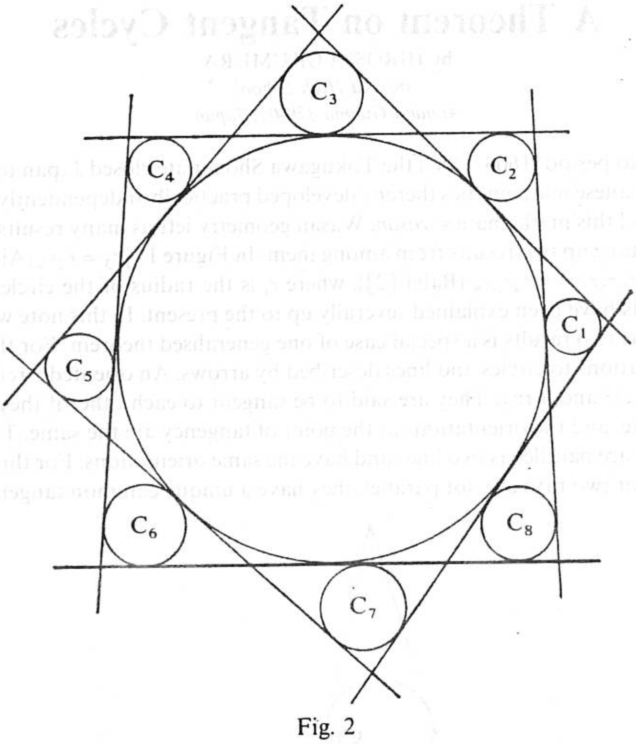

Fig. 2

will denote this cycle and its radius by $C_{x,y,z}$ and $r_{x,y,z}$, where we define the sign of $r_{x,y,z}$ plus if the orientation is anti-clockwise otherwise minus. For a ray x, $-x$ denotes the cycle along the same line with opposite orientation to x. For two rays x, y (x, y) is an oriented angle from x to y.

We need two lemmas.

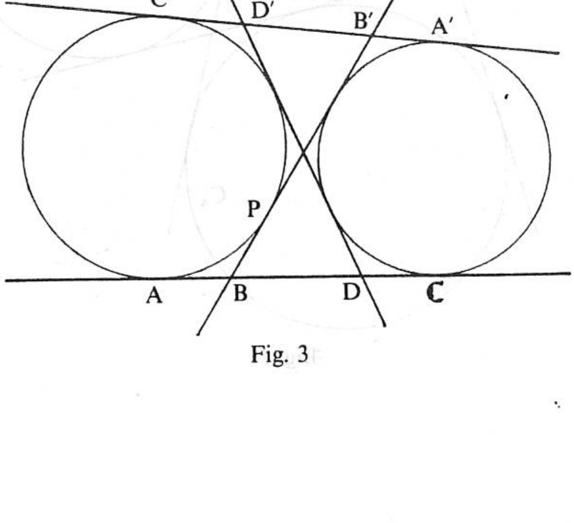

Fig. 3

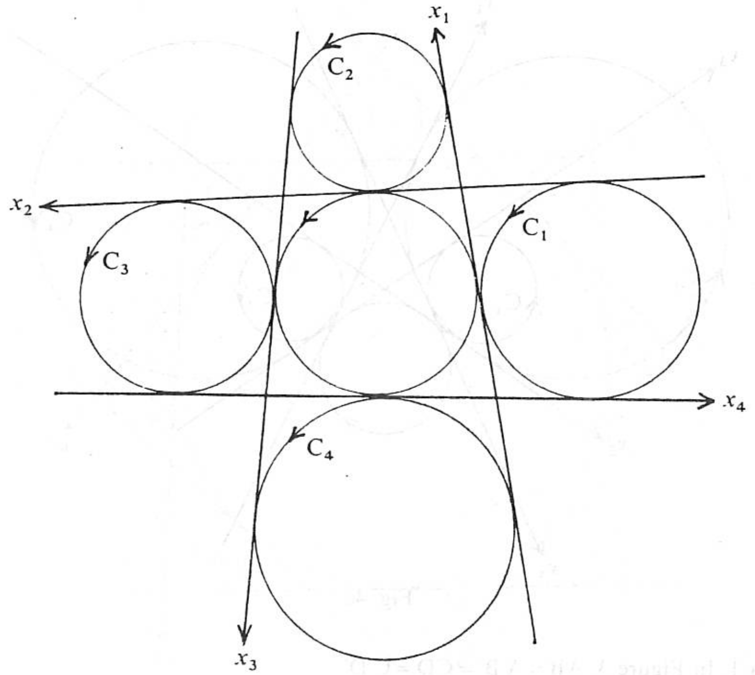

Fig. 4a

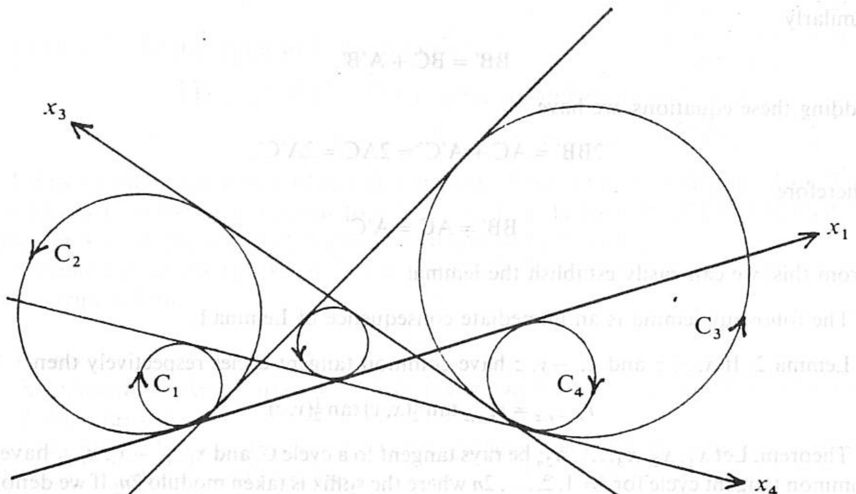

Fig. 4b

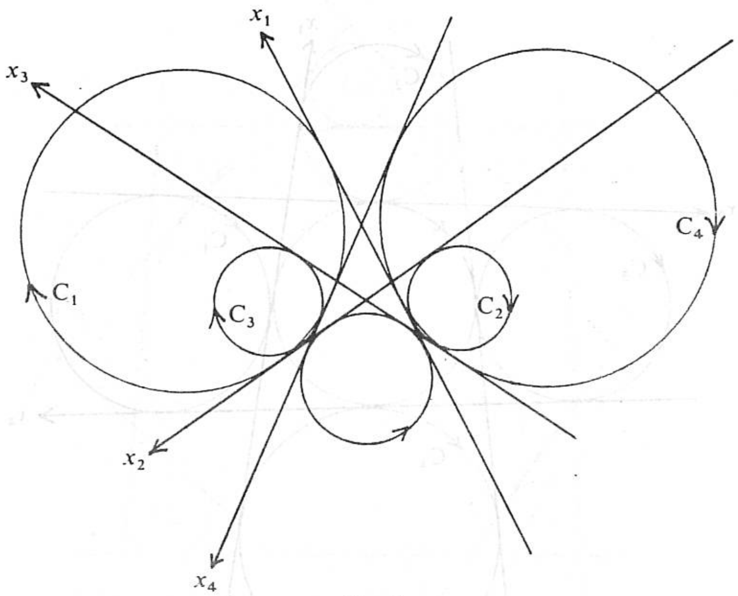

Fig. 4c

Lemma 1. In Figure 3, AB = A′B′ = CD = C′D′.

Proof. In the figure,

$$BB' = BP + PB' = AB + B'C'.$$

Similarly

$$BB' = BC + A'B'.$$

Adding these equations, we have

$$2BB' = AC + A'C' = 2AC = 2A'C'.$$

Therefore

$$BB' = AC = A'C'.$$

From this, we can easily establish the lemma.

The following lemma is an immediate consequence of Lemma 1.

Lemma 2. If x, y, z and x, $-y$, z have common tangent cycles respectively then

$$r_{x,-y,z} = r_{x,y,z} \tan\tfrac{1}{2}(x, y) \tan\tfrac{1}{2}(y, z).$$

Theorem. Let $x_1, x_2, x_3, \ldots, x_{2n}$ be rays tangent to a cycle C, and x_{i-1}, $-x_i$, x_{i+1} have a common tangent cycle for $i = 1, 2, \ldots, 2n$ where the suffix is taken modulo $2n$. If we denote the radius of this cycle by r_i then

$$r_1 r_3 \ldots r_{2n-1} = r_2 r_4 \ldots r_{2n}.$$

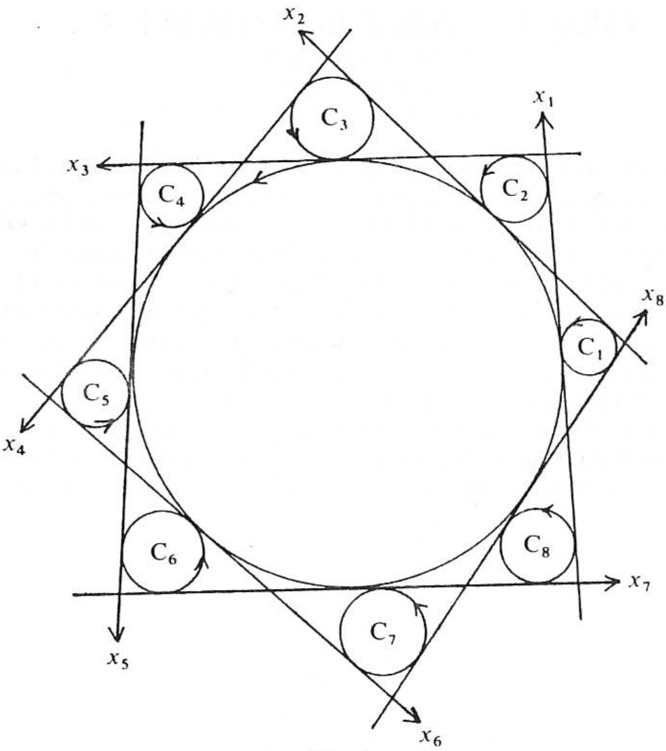

Fig. 5

Proof. Let r be the radius of C. By Lemma 2,

$$\begin{aligned}\prod r_{2k-1} &= r^n \prod \tan \tfrac{1}{2}(x_{2k-2}, x_{2k-1}) \tan \tfrac{1}{2}(x_{2k-1}, x_{2k}) \\ &= r^n \prod \tan \tfrac{1}{2}(x_{2k-1}, x_{2k}) \tan \tfrac{1}{2}(x_{2k}, x_{2k+1}) = \prod r_{2k}.\end{aligned}$$

Let us consider examples satisfying the condition of the theorem (see Figures 4 and 5). In the figures C_i is the cycle tangent to x_{i-1}, $-x_i$, x_{i+1}. Figures 4a and 5 show that the theorem is a generalisation of the two results mentioned above.

I would express my thanks to Dr Hirayama who kindly informed me where Baba's manuscript is kept.

References

1. Aida, Sampō Kokon Tsūran, 1797 (Catalogue number 1625, Japan Academy Library, Tokyo, Japan).
2. Baba, Jimon Jitō Daijutsu Vol. 1, 1816 (Catalogue number 1683, Tōhoku University Hayashi Library, Sendai, Japan).

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