Tiling with a Trilateral Trapezium and Penrose Tiles

Michael de Villiers

RUMEUS, University of Stellenbosch

profmd1@mweb.co.za

INTRODUCTION

Tilings have been around in human culture for hundreds if not thousands of years, mainly for decorative, religious and artistic purposes. However, tiling also has a functional component – consider for example a tiled kitchen or bathroom where it is far easier to replace a single broken tile than it would be to repair a wall or floor covered by one large sheet. In the past century or so tilings have been more rigorously investigated mathematically, but there are still a number of unanswered questions.

PENROSE TILES

The famous Penrose tiles are named after the mathematician and physicist Sir Roger Penrose, who investigated them in the 1970s. The Penrose kite is a quadrilateral which has three interior angles of 72° and one of 144°, while the Penrose dart is a concave quadrilateral which has two interior angles of 36°, one of 72° and one of 216°. Combined together they form a rhombus as shown in Figure 1.

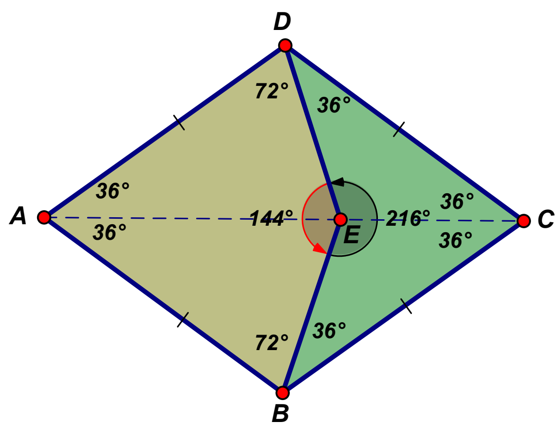

FIGURE 1: Penrose tiles.

An interesting mathematical property of Penrose tiles is that the sides are in the golden ratio, φ:

$$\frac{AD}{DE} = \frac{CD}{DE} = \frac{1+\sqrt{5}}{2}$$

Proving this is a straightforward exercise in trigonometry and is left to the reader.

When combined above to form a rhombus, it is clear that one can tile the plane quite easily with translations of this rhombus. This would produce a *periodic* tiling, i.e. a tiling that has *translation* symmetry. However, the specific matching rules for Penrose tiles prohibit the formation of such a rhombus. These matching rules can be achieved by using a pattern of circular arcs as shown in Figure 2 to constrain the placement of tiles: when two tiles share an edge in a tiling, the patterns must match at these edges, and not only the colors, but the arcs also have to match. An example of properly matched Penrose tiles is shown in Figure 3.

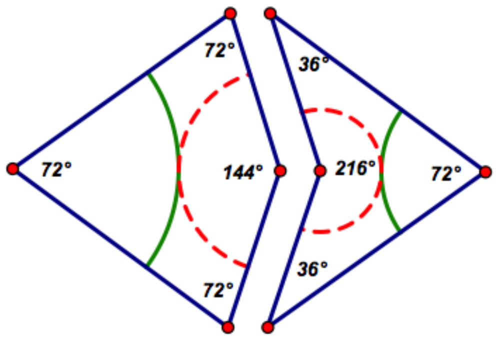

FIGURE 2: Penrose matching rules.

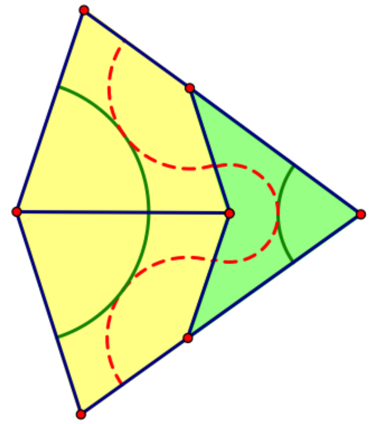

FIGURE 3: Properly matched Penrose tiles.

Using these matching rules one can now produce an *aperiodic* tiling which does not contain arbitrarily large periodic patches, i.e. a tiling which has no translation symmetry as shown in Figure 4.

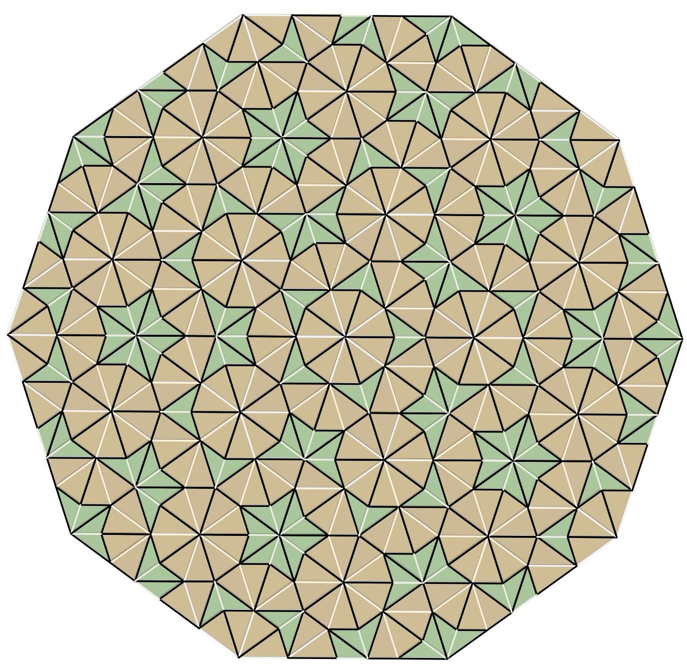

FIGURE 4: Aperiodic tiling of Penrose kites and darts.

TRIANGULAR KITES AND TRILATERAL TRAPEZIUM

The convex Penrose tile is a special case of what I've called a 'triangular kite' in De Villiers (2009), and also at this link to a hierarchical classification of quadrilaterals:

http://dynamicmathematicslearning.com/quad-tree-new-web.html

A 'triangular kite' can be defined as any kite with at least three equal *angles*. In similar vein, the 'trilateral trapezium', also mentioned and defined at the same reference and link above, can be defined as any isosceles trapezium with at least three equal *sides* (see Figure 5).

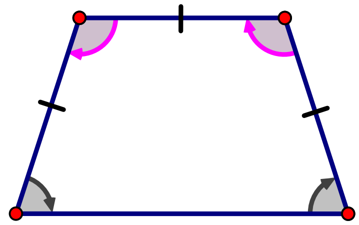

FIGURE 5: Trilateral trapezium.

The aesthetically pleasing side-angle duality between a triangular kite and a trilateral trapezium can neatly be contrasted as illustrated below.

TRIANGULAR KITE	TRILATERAL TRAPEZIUM
At least one axis of symmetry through a pair of opposite *angles* (vertices)	At least one axis of symmetry through a pair of opposite *sides*
(At least) two pairs of adjacent *sides* equal	(At least) two pairs of adjacent *angles* equal
(At least) three equal *angles*	(At least) three equal *sides*
Angle bisectors are concurrent at the *incentre;* i.e. has *inscribed circle*	Perpendicular bisectors of the *sides* are concurrent at the *circumcentre*; i.e. has *circumscribed circle*

What appears to be less well known is that one can produce pleasing periodic tilings of each of the Penrose tiles with its dual, a trilateral trapezium with two of its angles equal to 72° with the other two angles equal to 108°. It is again left to the reader to show that a 'golden trilateral trapezium' has its longest side also in the golden ratio to the three equal sides. Since Penrose tiles as well as this trilateral trapezium involve the golden ratio, they are also examples of what have been respectively called 'golden kites' and 'golden trapezia' in De Villiers (2017). Examples of these periodic tilings are illustrated in Figures 6 – 9.

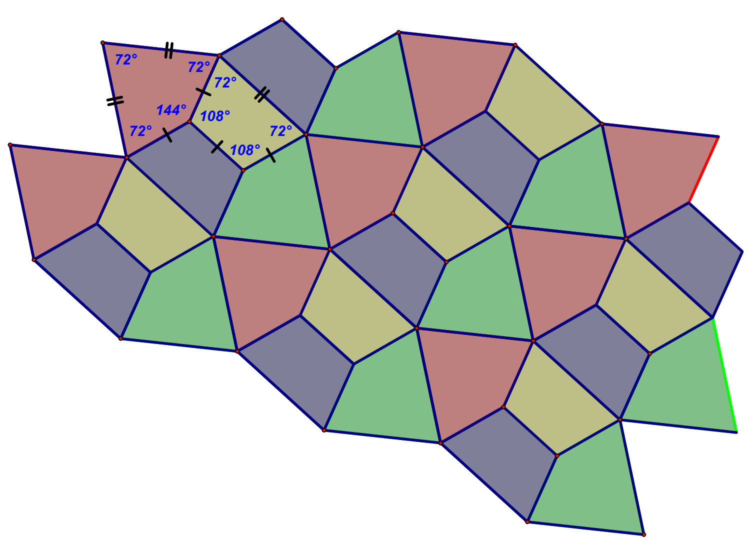

FIGURE 6: Example of periodic tiling.

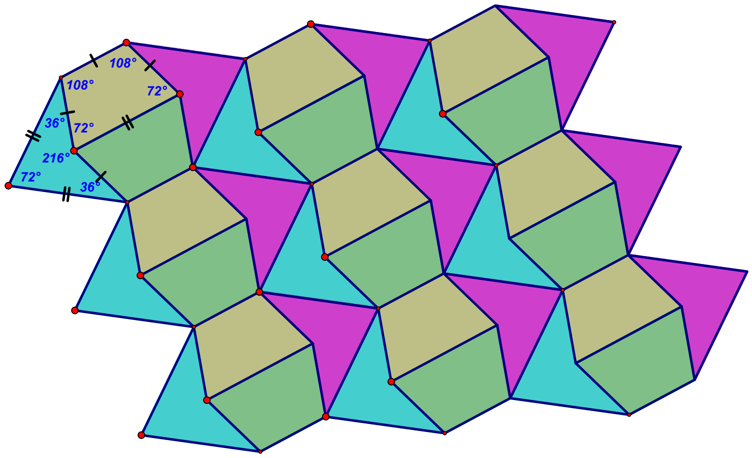

FIGURE 7: Example of periodic tiling.

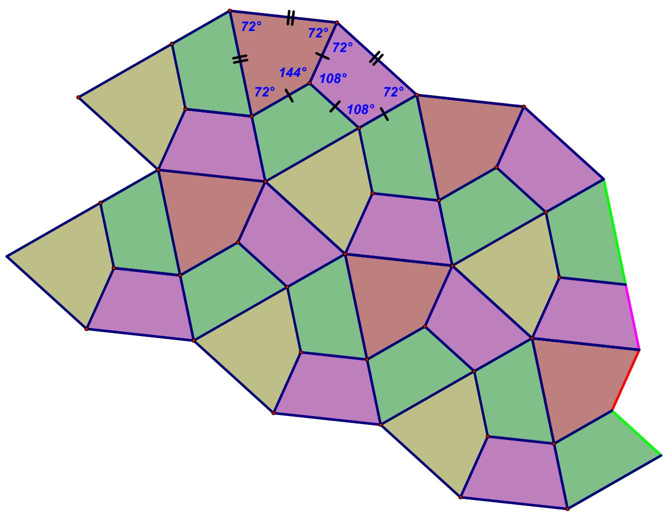

FIGURE 8: Example of periodic tiling.

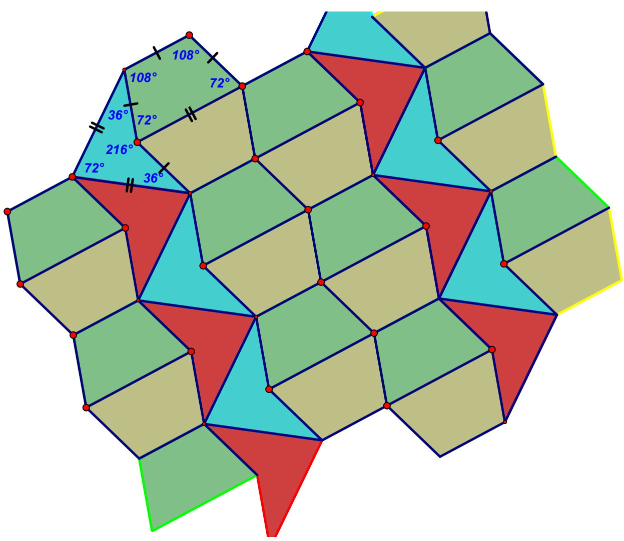

FIGURE 9: Example of periodic tiling.

Due to the properties of the golden trilateral trapezium and the Penrose tiles, they can fit together in many different ways. However, not all patterns one can produce will necessarily lead to a tiling of the plane, as illustrated in Figure 10.

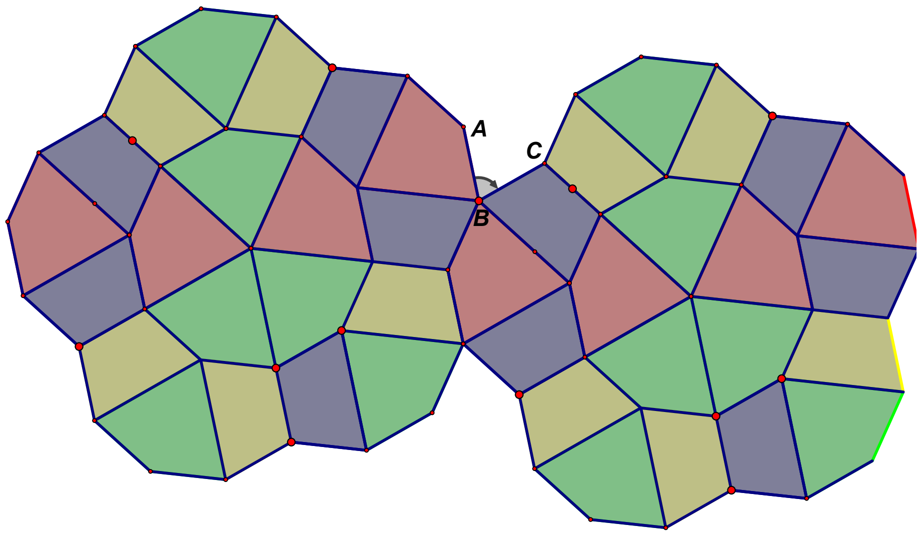

FIGURE 10: Non-tessellating pattern.

Although a pleasing design, the pattern shown in Figure 10, created with trilateral trapeziums and Penrose kites, does not tessellate. Since the angle *ABC* is equal to 72°, and *AB* = *CB*, one cannot correctly fit into this space either the golden trilateral trapezium or the Penrose kite.

Of course, the interesting mathematical question is whether one could define restrictive matching rules with a golden trilateral trapezium and one or both of the Penrose tiles to produce an aperiodic tiling.

CONCLUDING COMMENT

With tessellations prescribed in the GET phase in South Africa, learners could be challenged to create tilings like these with the Penrose tiles and the golden trilateral trapezium using cardboard tiles and/or dynamic geometry software. Perhaps one of them could discover a matching rule for producing a new aperiodic tiling!

REFERENCES

De Villiers, M. (2009). *Some adventures in Euclidean geometry*. Lulu Publishers, Raleigh, North Carolina.

De Villiers, M. (2017). An example of constructive defining: From a Golden Rectangle to Golden Quadrilaterals and beyond. *At Right Angles*, Vol. 6, No. 1, pp. 64-69; Vol. 6, No. 2, pp. 74-81.