Toshio Seimiya's hexagon theorem

In a plane hexagon ABCDEF, the lines joining the midpoints of the opposite sides are concurrent if, and only if, triangles ACE and BDF have equal areas.

The theorem as stated holds for convex and concave hexagons, but can fail for some crossed hexagons. To be true generally, the triangles must be described *in the same sense,* i.e. both must be clockwise or both anticlockwise. In other words their *signed* areas must be equal. We revise the statement of the theorem, building in this idea and starting with the triangles.

The coplanar triangles ACE and BDF are described in the same sense and have equal areas. Prove that the lines joining the midpoints of opposite sides of the hexagon ABCDEF are concurrent, and show that the converse is true.

Let the midpoints of *AB, BC, CD, DE, EF, FA* be *P, Q, R, S, T, U* respectively. We are to prove that *PS, QT* and *RU* are concurrent. There is no loss of generality if we assume that both triangles are anticlockwise.

First we simplify the diagram, keeping its relevant properties. Draw ΔACE, and translate ΔBDF by a vector **BA**. The vertex *B* now coincides with *A.* The six midpoints are translated by ½**BA**, as are the lines *PS, QT, RU* and their intersections. If these lines are concurrent in the original diagram, they are concurrent in the transformed version.

The quicker way now is to set up oblique axes *AC* and *AE*, with scales such that $C = (2, 0)$ and $E = (0, 2)$. We calculate areas in the usual way—they are in arbitrary units. So area *ACE* = 2 units.

Alternatively we can use two more transformations. Geometrical one-way stretches and shears map lines on to lines, mid-points on to midpoints, and equal areas on to equal areas. We keep *AC* fixed, shearing the figure in a direction parallel to *AC* until *AE* is perpendicular to *AC*. Then we apply a one-way stretch (or squash) that makes $AE = AC$, leaving *AC* and $\angle CAE$ unchanged. In this new diagram we take *A* as (0, 0), *C* as (2, 0) and *E* as (0, 2). (The '2's help us avoid unnecessary fractions in the working.) With either method the rest of the working is the same.

Since *B* coincides with *A* it is also (0, 0). Let *D* be $(2j, 2k)$ and *F* be $(2l, 2m)$. The standard formula for a triangular area gives $\triangle BDF = ½\{x_B (y_D – y_F) + x_D (y_F – y_A) + x_F (y_A – y_D)\} = 2(jm – kl)$. This is positive if, and only if, *B—D—F—B* is anticlockwise. Note that the vertices *D* and *F* need not be in the first quadrant: any of *j, k, l* and *m* can be zero or negative.
Also $\triangle ACE$ is anticlockwise, and its signed area is +2 units. Since the two areas are equal, we have $jm – kl = 1$.

The six midpoints are $P(0, 0)$, $Q(1, 0)$, $R(j+1, k)$, $S(j, k+1)$, $T(l, m+1)$ and $U(l, m)$. Hence the equations of the lines *PS, QT* and *RU* are:
$(1 + k)x – j y = 0$; $(–m – 1)x + (l – 1)y = –(1 + m)$; and $(m – k)x + (1 + j – l)y = m(1 + j) – lk$.

For these to be concurrent, the equations must be consistent. Now if we add all three, the terms in *x* and *y* vanish, so for consistency the sum of the right-hand sides, $–1 + jm – lk$, must also vanish. But we have seen that $jm – lk = 1$, so $–1 + jm – lk = 0$. The three equations are therefore consistent, hence the lines *PS, QT* and *RU* are concurrent.

Finally, we undo the initial translation, restoring *BDF* to its original position. The three lines are translated into their correct position, and are concurrent.

The converse is obvious.

Michael Fox
March 2019