

## Border Patterns, Tessellations & Problem Solving with Transformations

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*This paper will firstly explore some of the symmetries and transformations involved in border patterns and tessellations. This will be followed by some examples of problem solving with transformations. Lastly, we'll explore some transformations of graphs of functions of the form  $y = f(x)$ .*

### Introduction

The new national curriculum statement (Dept. of Ed., 2002a) specifically describes the following outcomes for the Senior Phase (Grades 7-9):

- \* Use national flags to demonstrate transformations and symmetry in designs.
- \* Investigate and appreciate the geometrical properties and patterns in traditional and modern architecture (e.g. construction and painting of Ndebele homes).
- \* Use maps in geography as specific forms of grids.
- \* Investigate geometric patterns in art (e.g. African and Islamic art).

For the FET phase (Grades 10-12) the following outcomes are specified in the Guidelines for Learning Programmes (Dept. of Ed., 2002b):

- *develops conjectures and generalisations, related to triangles, quadrilaterals and other polygons, and attempts to validate, justify, explain or prove them, using any logical method (Euclidean, co-ordinate and transformation)*
- *generates as many graphs as necessary by means of point by point plotting to test conjectures and hence generalise the effects of parameters on the graphs of functions*

### Border patterns

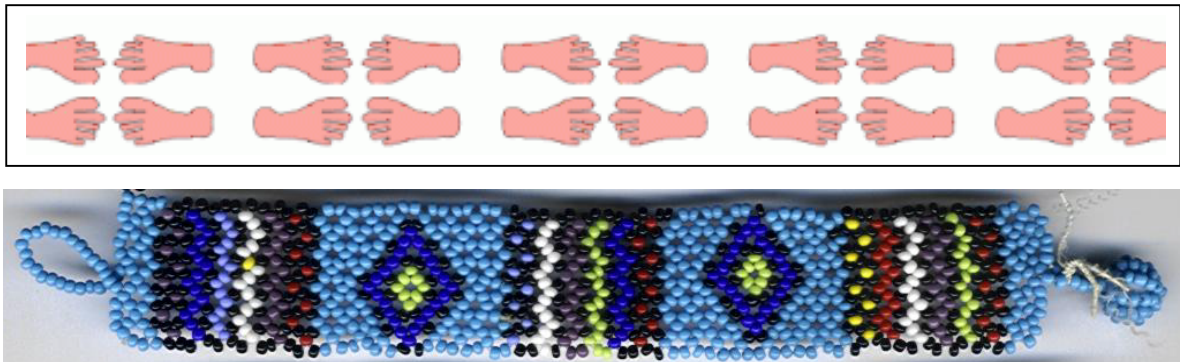
Border or frieze patterns are one dimensional, repeating patterns that are often used as decoration on the EDGES of garments, books, buildings, plates, rugs, etc. There are seven types of border patterns based on their symmetry properties of reflection, translation and rotation.

The classification system normally used assigns to each border pattern a two letter/number code to label its type. By definition it is



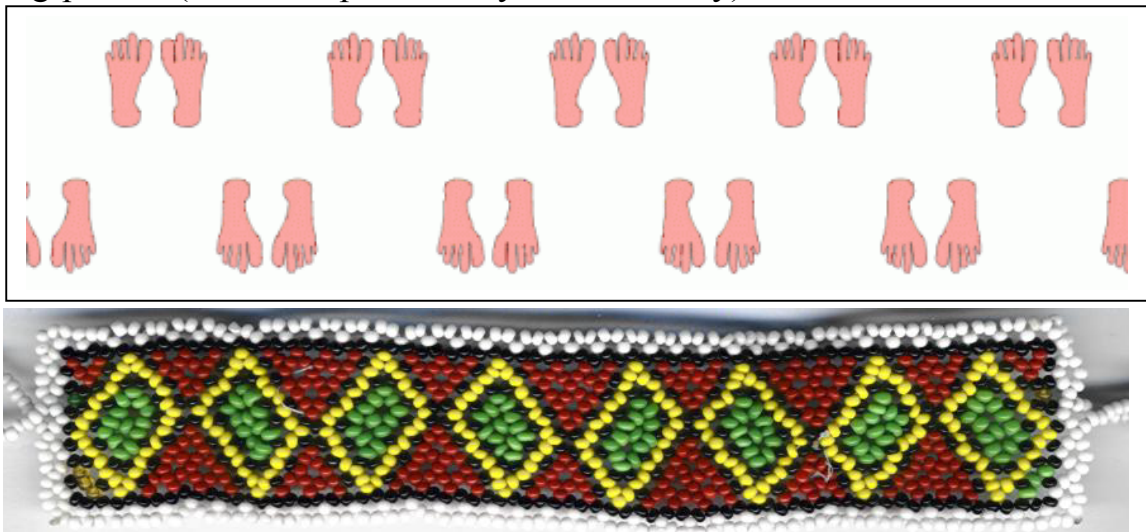
(Note that if we completely ignore the colours in the second example, it is classified as a mg-pattern.)

mm-pattern (called a ‘spinjump’ by John Conway)



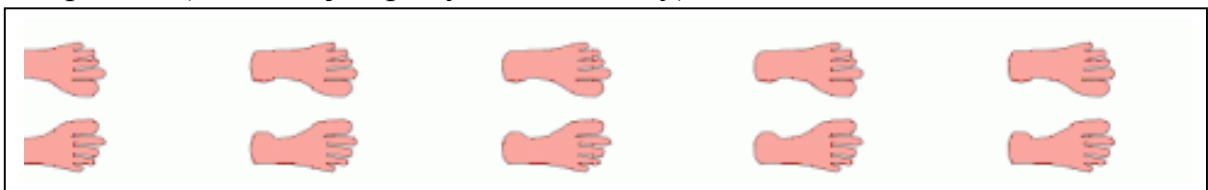
A mm-pattern has vertical and horizontal line symmetries in addition to its translation symmetry. (Note that in classifying the second example as mm above, the colours of the vertical lines were ignored, which gives the same classification as if all the colours are ignored. But with the colours of the vertical lines as given, it would actually be a 1m-pattern, because the colouring of the vertical lines does not have vertical line symmetry.)

mg-pattern (called a ‘spinside’ by John Conway)

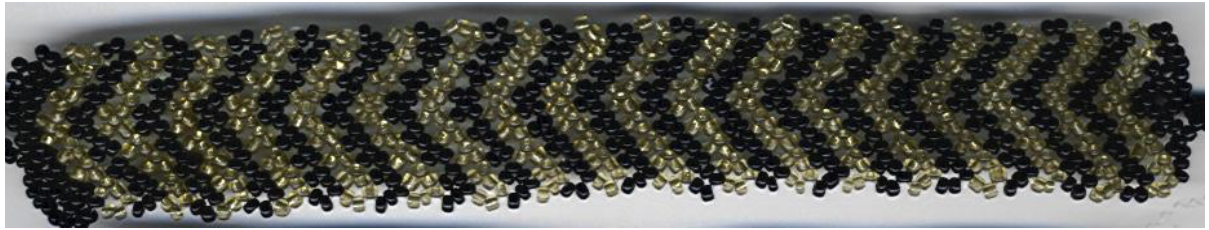


A mg-pattern has vertical and horizontal glide reflection symmetries in addition to its translation symmetry.

1m-pattern (called a ‘jump’ by John Conway)







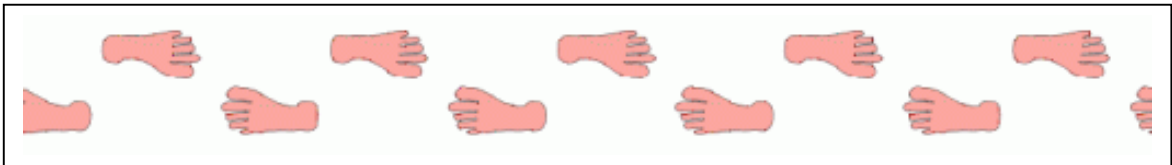
A 1m-pattern has horizontal reflection symmetry in addition to its translation symmetry.

1g-pattern (called a 'step' by John Conway)



A 1g-pattern has horizontal glide reflection symmetry in addition to its translation symmetry. This is not a very common pattern in traditional African art.

12-pattern (called a 'spinhop' by John Conway)



A 12-pattern has half-turn symmetry in addition to its translation symmetry. (Note that if we completely ignore the colours in the second example, it is classified as a mg-pattern.)

## Tessellations

A tessellation is created when a shape is repeated over and over again covering a plane without any gaps or overlaps.

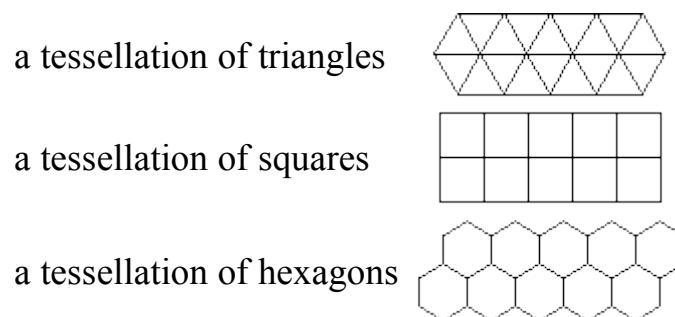
Another word for a tessellation is a **tiling**.

A dictionary will tell you that the word "tessellate" means to form or arrange small squares in a checkered or mosaic pattern. The word "tessellate" is derived from the Ionic version of the Greek word

"tesserer," which in English means "four." The first tilings were made from square tiles.

A regular polygon has 3 or 4 or 5 or more sides and angles, all equal. A **regular tessellation** means a tessellation made up of congruent regular polygons. [Remember: *Regular* means that the sides of the polygon are all the same length. *Congruent* means that the polygons that you put together are all the same size and shape.]

Only three regular polygons tessellate in the Euclidean plane: triangles, squares or hexagons. We can't show the entire plane, but imagine that these are pieces taken from planes that have been tiled. Here are examples of



When you look at these three samples you can easily notice that the squares are lined up with each other while the triangles and hexagons are not. Also, if you look at 6 triangles at a time, they form a hexagon, so the tiling of triangles and the tiling of hexagons are similar and they cannot be formed by directly lining shapes up under each other - a slide (or a glide!) is involved.

You can work out the interior measure of the angles for each of these polygons:

Shape	Angle measure in degrees
triangle	60
square	90
pentagon	108
hexagon	120
more than six sides	more than 120 degrees

Since the regular polygons in a tessellation must fill the plane at each vertex, the interior angle must be an exact divisor of 360 degrees. This works for the triangle, square, and hexagon, and you can show working

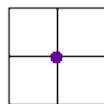
tessellations for these figures. For all the others, the interior angles are not exact divisors of 360 degrees, and therefore those figures cannot tile the plane.

### Naming Conventions

A tessellation of squares is named "**4.4.4.4**". Here's how: choose a vertex, and then look at one of the polygons that touches that vertex. How many sides does it have?

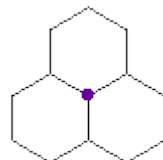
Since it's a square, it has four sides, and that's where the first "4" comes from. Now keep going around the vertex in either direction, finding the number of sides of the polygons until you get back to the polygon you started with. How many polygons did you count?

There are four polygons, and each has four sides.



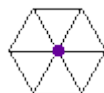
4.4.4.4

For a tessellation of regular congruent hexagons, if you choose a vertex and count the sides of the polygons that touch it, you'll see that there are three polygons and each has six sides, so this tessellation is called "**6.6.6**":



6.6.6

A tessellation of triangles has six polygons surrounding a vertex, and each of them has three sides: "**3.3.3.3.3.3**".



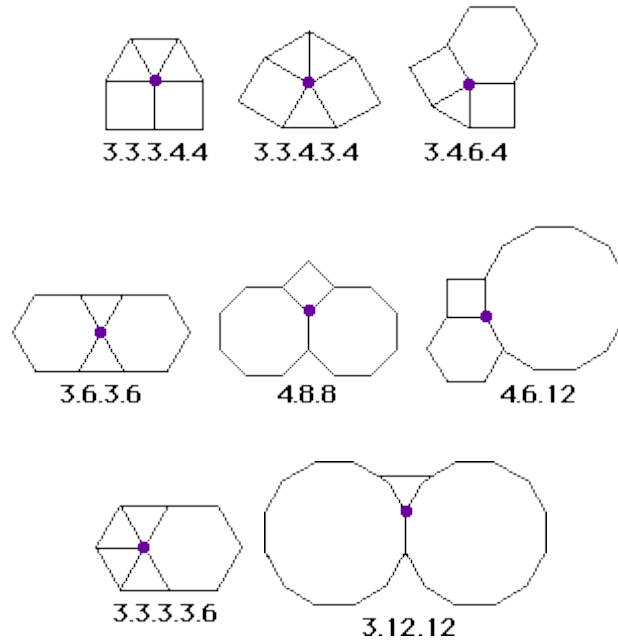
3.3.3.3.3.3

### Semi-regular Tessellations

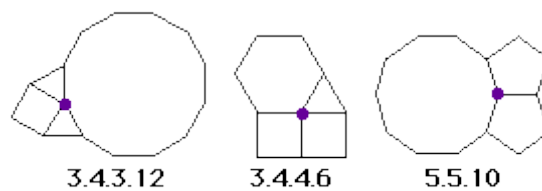
You can also use a variety of regular polygons to make **semi-regular tessellations**. A semi-regular tessellation has two properties, which are:

1. It is formed by regular polygons.
2. The arrangement of polygons at every vertex point is identical.

Here are the **eight** semi-regular tessellations:



Interestingly there are other combinations that seem like they should tile the plane because the arrangements of the regular polygons fill the space around a point. For example:



**Task 1:** If you try tiling the plane with these units of tessellation you will find that they cannot be extended infinitely. Fun is to try this yourself.

1. Hold down on one of the images and **copy** it to the clipboard.
2. Open a paint program.
3. Paste the image.
4. Now continue to paste and position and see if you can tessellate it.

**Task 2:** Explore the symmetries (translation, reflection, glide reflection & rotation) of the regular and semi-regular tessellations. How are they the same or different?

*Note:* There are an infinite number of tessellations that can be made of patterns that do not have the same combination of angles at every vertex point. There are also tessellations made of polygons that do not share common edges and vertices.

## Problem Solving with Transformations

Many real world problems can be easily modeled with *Sketchpad* and elegantly solved by using transformations such as reflections, translations and rotations. Below are some examples that will be briefly discussed.

### The Burning House Problem

A man is walking in an open field some distance from his house. It's a beautiful day and he is carrying an empty bucket with him to collect berries. Before long, he turns around and, to his horror, sees that his house on fire. Without wasting a moment, he runs to a nearby river (which runs in a straight line from east to west) to fill the bucket with water so that he can run to his house to throw water on the fire. Naturally, he wants to do this as quickly as possible. Describe how to construct the point on the river bank to which he should run in order to minimize his total running distance (and time)

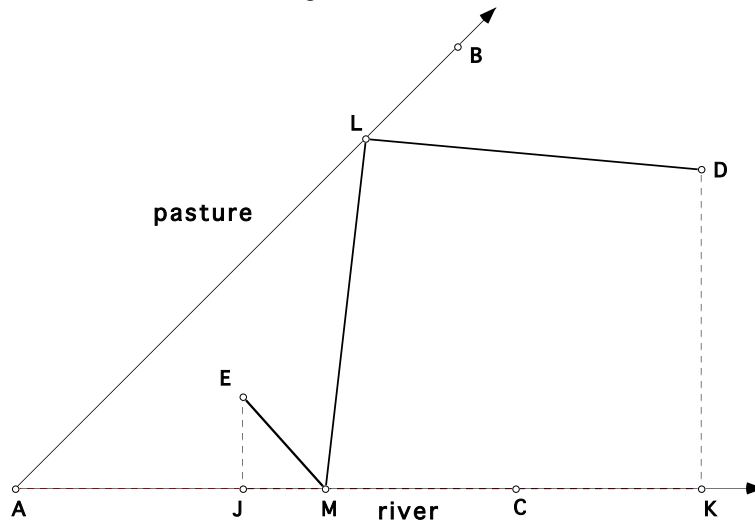


RIVER

### Horse Riding

A rider is traveling from point D to point E between a river and a pasture. Before she gets to E, she wants to stop at the pasture to feed her horse, and again at the river to water him. If angle  $BAC = 45^\circ$ ,  $EJ = 2$  km,  $AJ = 5$  km,  $DK = 7$  km and  $JK = 10$  km, what path should she take to travel the shortest possible distance?

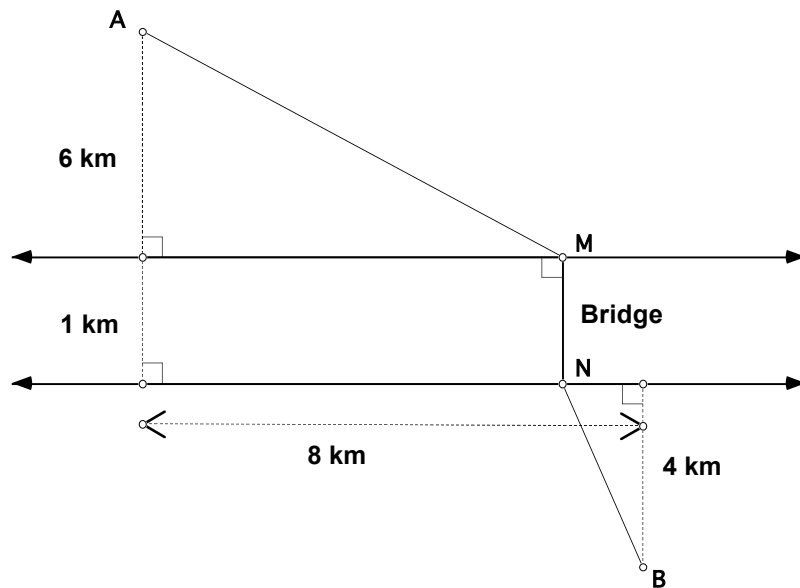




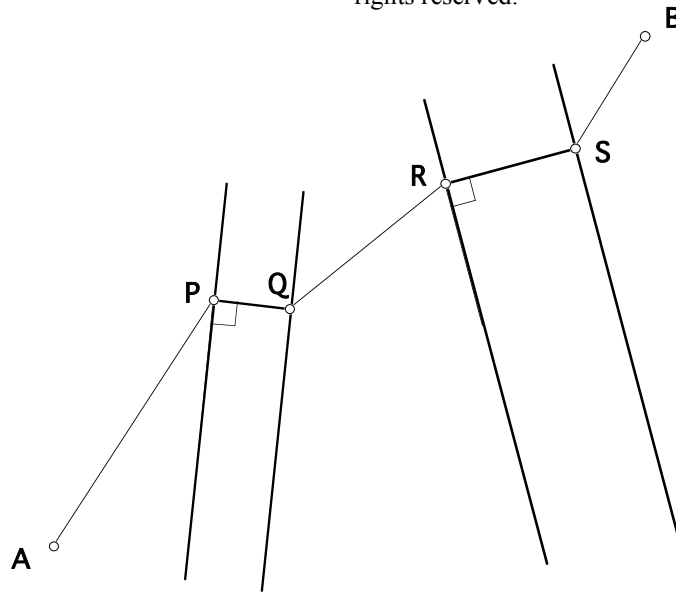
### Building Bridges

The following two problems are from Makae et al (2001).

1. At what point should a bridge MN be built across a river separating two towns A and B so that the path AMNB is as short as possible? (It is assumed the river consists of two parallel lines with the bridge perpendicular to it.)



2. Solve the same problem as in Question 1 if the towns A and B are separated by two rivers across which bridges PQ and RS have to be constructed (see above).

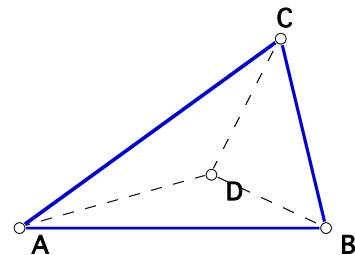


### Building an airport

The following problem is from De Villiers (2003).

Suppose an airport is planned to service three cities of more or less equal size. The planners decide to locate the airport so that the sum of the distances to the three cities is a minimum. Where should the airport be located?

DC = 2.006 cm  
 DB = 1.663 cm  
 DA = 2.653 cm  
 DC + DB + DA = 6.321 cm



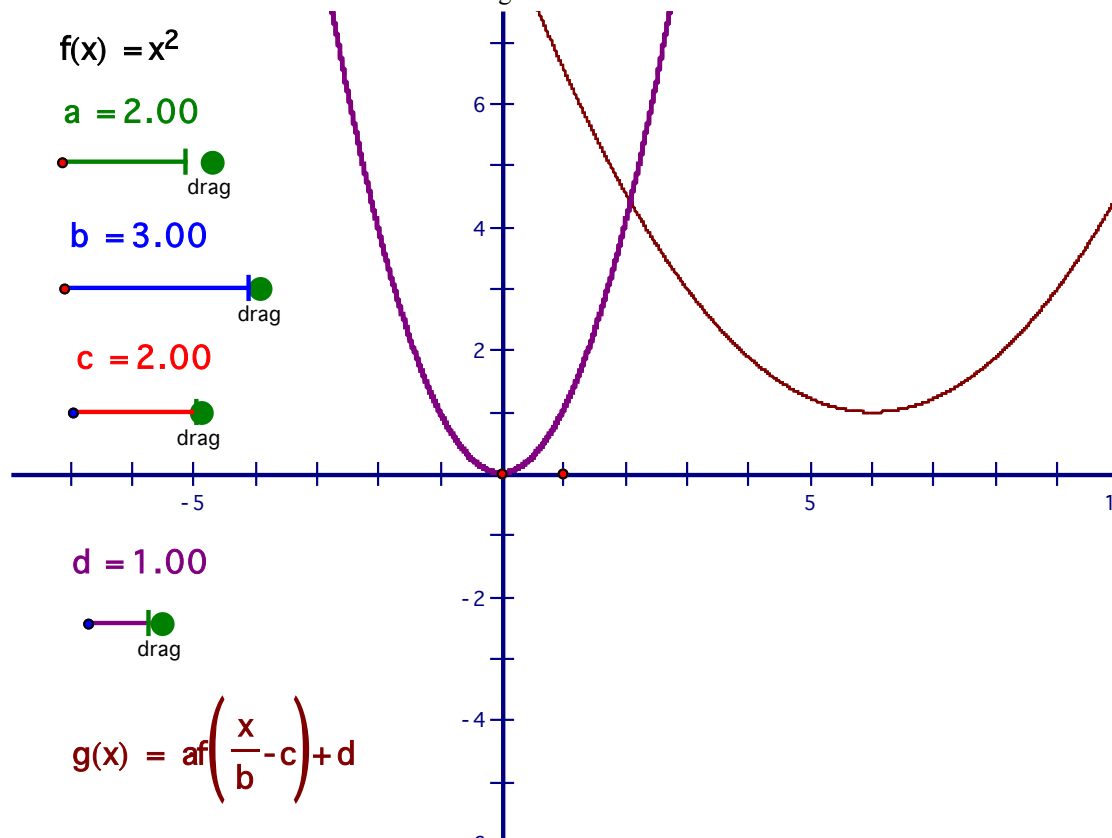
### Transformations of $y = f(x)$

As discussed in De Villiers (1991), the study of transformations forms a golden thread linking together many diverse parts of the mathematics curriculum, and beautifully connects geometry with algebra. Particularly interesting and relevant is to investigate different transformations of functions of the type  $y = f(x)$  with a dynamic tool like *Sketchpad*.

Essentially, there are two different, though related questions one could explore:

- (1) How can we transform a function  $y = f(x)$  in a particular way? (For example, how can we reflect it around the  $y$ -axis?)
- (2) What happens if we transform  $y = f(x)$  in a particular way? (For example, what happens to  $y = f(x)$  with the transformation  $g(x) = f(x) + b$ ?)

We shall use the *Sketchpad* sketch below to find transformations of  $y = f(x)$  for each of the following transformations:



Reflection in  $x$ - and  $y$ -axes  
 Reflection in  $y = x$   
 Half-turn around origin  
 Enlargement from origin  
 Stretch in  $x$ - and  $y$ -directions

It should be noted that in exploring and finding such transformations that one should not (just) use straight-line graphs as these can easily lead to incorrect generalizations, but use sufficiently general or a variety of functions.

## References

- Dept. of Education. (2002a). **Mathematics Learning Area Statements**, Version 4, Grades 1-9. Pretoria.
- Dept. of Education. (2002b). **Mathematics: Guidelines for Learning Programmes**, Draft, Grades 10-12. Pretoria.
- De Villiers, M. (1991). Transformations: A golden thread in school mathematics. **Spectrum**, 34(4), pp. 11-18. (Available at <http://mysite.mweb.co.za/residents/profmd/transform.pdf>)
- De Villiers, M. (2003). **Rethinking Proof with Sketchpad 4**. Key Curriculum Press, USA.
- Makae, E.J, Nhlapo, M. et al. (2001). **Let's Talk Mathematics!** (Grade 9 Mathematics Textbook), Juta-Gariep Publishers.