techspace

A Trapezium Theorem Generalized

"… the deductive method starting from seemingly dogmatic axioms provides a shortcut for covering large territory. But the constructive Socratic method that proceeds from the particular to the general and eschews dogmatic compulsion leads the way more surely to independent productive thinking." – Richard Courant (1964, p. 43).

Michael de Villiers

A nice investigation with dynamic geometry for students at high school is the so-called *Midpoint Trapezium theorem*, which appears in many popular geometry textbooks such as Serra (2008, pp. 276-277). It can be stated as follows: 'Given any trapezium *ABCD* with *AD* // *BC*, and if *E* and *F* are the respective midpoints of *AB* and *CD*, then *EF* is parallel to the other two sides *AD* and *BC*, and equal to half of their sum.'

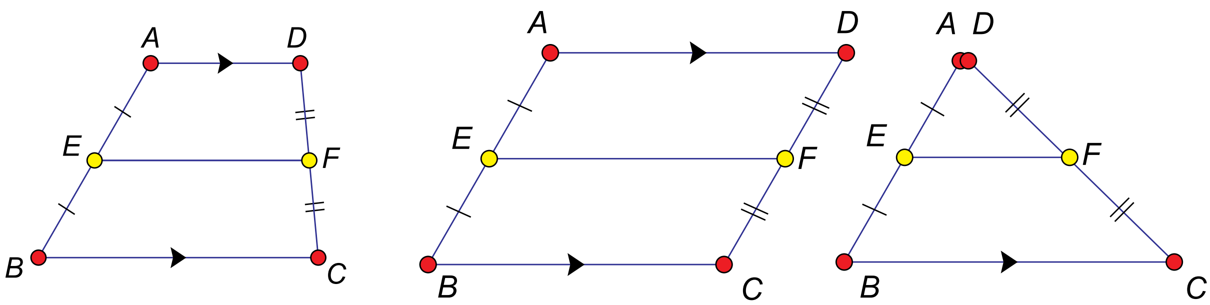

Figure 1

***Keywords**: Trapezium (trapezoid), half-turn, parallelogram, triangle inequality, translation, midpoint triangle theorem.*

With the aid of dynamic geometry, children can easily demonstrate this as indicated in Figure 1; and by dragging *D* they can show that the theorem applies to a parallelogram as a special case as well as to a triangle in the degenerate case, giving us the Midpoint Triangle theorem as another special case.

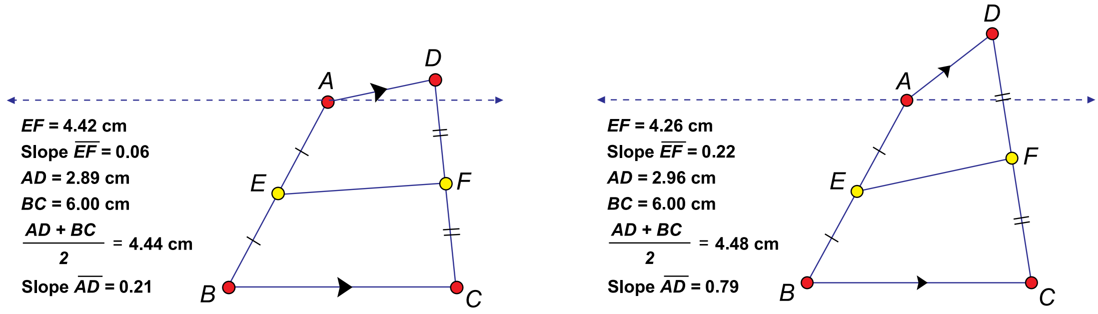

Figure 2

Conjecturing

It seems natural to ask: What happens if *ABCD* is not a trapezium? Instead of having to construct a new quadrilateral, and redo all the measurements and calculations, a useful feature of *Sketchpad*, not present in most other dynamic geometry programmes, is that one can simply select the command "Split point *D* from the line" through *A* parallel to *BC*, to obtain Figure 2 for a general quadrilateral *ABCD*.

Apart from *EF* obviously being no longer parallel to the other two sides, it is apparent and easily checked by repeated experimentation that *EF* is less than or equal to half the sum of *AD* and *BC*, with equality just when *AD // BC*. So by asking a simple question and then exploring it with dynamic geometry, we've obtained a nice, new conjecture (which as far as I've been able to ascertain so far seems to be new, or at the very least, not well-known).

Before going on, readers are now invited to first explore and convince themselves of the truth of this conjecture by using a dynamic, interactive sketch online, by dragging any of the vertices of *ABCD* at this URL: http://dynamicmathematicslearning.com/trapezium-theorem-generalized.html

Explaining

But **why** is the result true? Note that I'm NOT asking whether it is true, as one can easily obtain sufficient conviction through experimentation to answer that question. What is lacking in this case is *insight* and *understanding*: in other words, not a proof to verify it, but a proof to *logically explain it* (compare Hersh, 1993; De Villiers, 1997).

As pointed out by Pólya (1946) and others, a useful strategy in problem solving is often to look at the special case first, as that may give insight into why the general result is true (and thus help one to construct a deductive argument). So how can we explain (prove) the original trapezium result stated at the beginning?

A simple transformation approach might come to mind as follows. Give the trapezium *ABCD* a half-turn around the midpoint *F* as shown in Figure 3. From the properties of a half-turn it immediately follows that the formed quadrilateral *ABA´B´* is a parallelogram, for which the result is intuitively and visually obvious. To formally prove this and understand why that is the case for the parallelogram, note that *A* and *B* can both be translated by the same vector *AB´* to respectively map onto *B´* and *A´*; but the same translation by vector *AB´* maps midpoint *E* onto midpoint *E´* since a translation preserves distance, direction and segment-length; hence *EF* is parallel to the other two sides and $2EF = (AD + DB') \Rightarrow EF = (AD + BC)/2$.

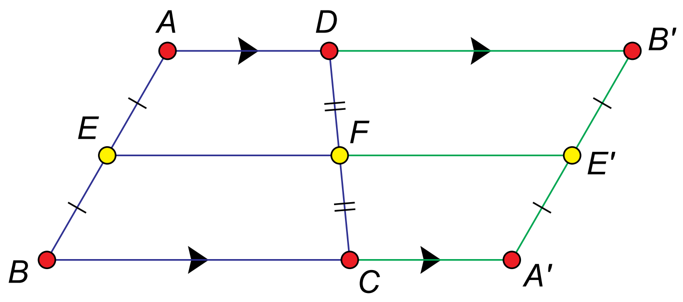

Figure 3

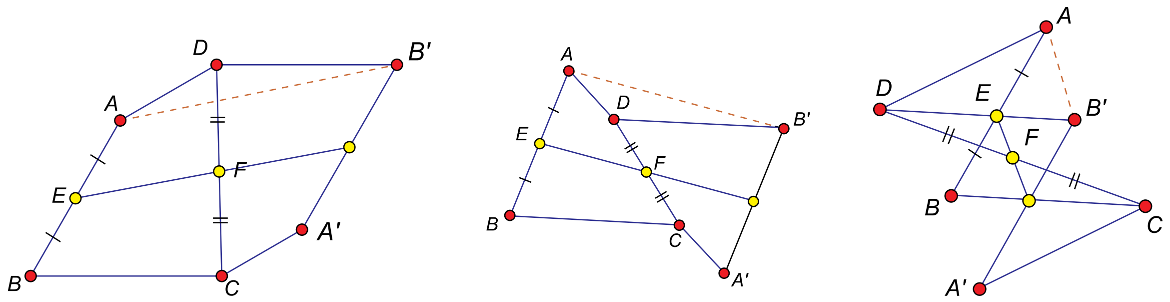

Figure 4

But what happens when *ABCD* is not a trapezium, but a general convex, concave or crossed quadrilateral? As shown in Figure 4, a half-turn around *F* produces a parallelo-hexagon (a hexagon with opposite sides equal and parallel), and therefore as before *ABA´B´* is still a parallelogram. Therefore, $EF = AB' / 2$. But from the triangle inequality $AB' < AD + DB' = AD + BC$, and this concludes the proof.

So the general inequality $EF \leq (AD + BC) / 2$ is merely a straightforward consequence of the triangle inequality, and nicely explains why the result is true. The traditional Euclidean proof for the Midpoint Trapezium theorem relies on the midpoint triangle theorem by drawing a diagonal *AC* with its midpoint *G* as shown in the first sketch in Figure 5 (compare Kay, 1994: 220; Alexander & Koeberlein, 2007: 209). But as shown for the convex case when *ABCD* is not a trapezium in Figure 5, the general inequality again easily follows from the triangle inequality. It is left to the reader to fill in the details and to check the concave and crossed cases as well.

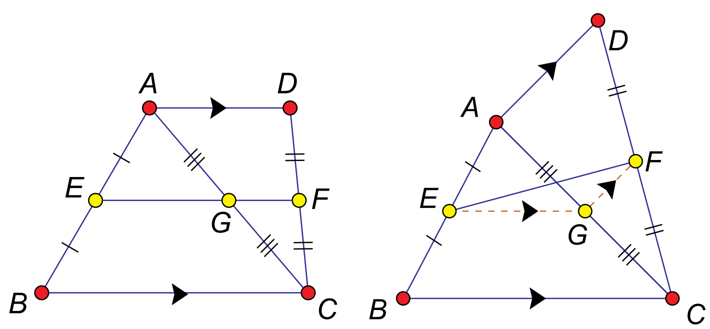

Figure 5

Conclusion

This extension of the Midpoint Trapezium theorem can be used to engage students using dynamic geometry in observing and making an interesting further conjecture, well within their means to logically explain (prove) with the triangle inequality theorem.

References

[1] Alexander, D.C. & Koeberlein, G.M. (2007). *Elementary Geometry for College Students*. Boston: Houghton Mifflin Company.

[2] Courant, R. (1964). Mathematics in the Modern World. *Scientific American*, Sept., Vol. 211, no. 3, pp. 41-49.

[3] De Villiers, M. (1997). The Role of Proof in Investigative, Computer-based Geometry: Some personal reflections. Chapter in Schattschneider, D. & King, J. (1997).*Geometry Turned On!* Washington: MAA, pp. 15-24.

[4] Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics*, 24, pp. 389-399.

[5] Kay, D. (1994). *College Geometry: A discovery approach*. New York: Harper Collins Publishers.

[6] Polya, G. (1945). *How to solve it*. Princeton: Princeton University Press.

[7] Serra, M. (2008, Edition 4). *Discovering Geometry: An Investigative Approach*. Emeryville: Key Curriculum Press.

MICHAEL DE VILLIERS has worked as researcher and mathematics and science teacher at several institutions across the world. Since 1991 he has been part of the University of Durban-Westville (now the University of KwaZulu-Natal). He has already published 9 books and 195 reviewed articles. He was editor of Pythagoras, the research journal of the Association of Mathematics Education of South Africa (AMESA) and has been vice-chair of the SA Mathematics Olympiad since 1997. He is a regular speaker at local and international conferences on mathematics and mathematics education. His main research interests are Geometry (especially Dynamic Geometry), Proof, Applications and Modeling, Problem Solving, and the History and Philosophy of Mathematics.

Prof de Villiers may be contacted on profmd@mweb.co.za. His home page is http://dynamicmathematics-learning.com/homepage4.html. He maintains a web page for dynamic geometry sketches at http://dynamic-mathematicslearning.com/JavaGSPLinks.htm.