Chapter 2
Geometry and Trigonometry

2.1 Geometric Inequalities

One of the most basic geometric inequalities is the triangle inequality: in every triangle, the length of one side is less than the sum of the two other sides' lengths. More generally, for any three points A, B, C one has

$$AC + BC \geq AB$$

with equality if and only if C lies on the line segment AB. Many interesting problems can be solved using this simple idea.

Problem 2.1 Let $ABCD$ be a convex quadrilateral and let M, N be the midpoints of AD and BC, respectively. Prove that

$$MN = \frac{AB + CD}{2}$$

if and only if AB is parallel to CD.

Solution Let P be the midpoint of the diagonal AC (see Fig. 2.1). Then MP and PN are parallel to CD and AB, respectively. Moreover, we have $MP = \frac{CD}{2}$ and $PN = \frac{AB}{2}$. Applying the triangle inequality to $\triangle MNP$ gives

$$\frac{AB}{2} + \frac{CD}{2} = PN + MP \geq MN.$$

The equality occurs if and only if P lies on the line segment MN; that is, if the lines MP, PN and MN coincide. Since $MP \parallel CD$ and $NP \parallel AB$, the latter holds true if and only if AB and CD are parallel.

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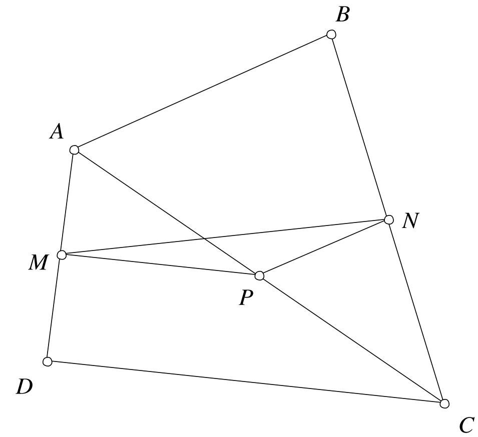
Fig. 2.1



Problem 2.2 Let D be the midpoint of the side BC of triangle ABC. Prove that

$$AD < \frac{AB + AC}{2}.$$

Solution Let the point A' be such that D is the midpoint of AA'. Thus, the quadrilateral $ABA'C$ is a parallelogram and $AD = \frac{AA'}{2}$. In triangle ABA', we have $AA' < AB + BA'$. The conclusion follows observing that $BA' = AC$.

Problem 2.3 Let M be a point inside the triangle ABC. Prove that

$$AB + AC > MB + MC.$$

Solution Let N be the point at which BM intersects AC. Then we successively have

$$AB + AC = AB + AN + NC > BN + NC = BM + MN + NC > BM + MC.$$

Problem 2.4 Let A, B, C and D be four points in space, not in the same plane. Prove that

$$AC \cdot BD < AB \cdot CD + AD \cdot BC.$$

Solution Consider a sphere which passes through the points B, C and D and intersects the segments AB, AC and AD at the points B', C' and D'. The intersection of the sphere and the plane (ABC) is a circle, thus the quadrilateral $BB'C'C$ is cyclic. It follows that triangles ABC and $AC'B'$ are similar; hence

$$\frac{AB'}{AC} = \frac{AC'}{AB} = \frac{B'C'}{BC}.$$

We obtain

$$B'C' = \frac{BC \cdot AB'}{AC}.$$