International Journal of Mathematical Education in Science and Technology

ISSN: 0020-739X (Print) 1464-5211 (Online) Journal homepage: http://www.tandfonline.com/loi/tmes20

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To cite this article: Michael de Villiers (2016): An explanatory, transformation geometry proof of a classic treasure-hunt problem and its generalization, International Journal of Mathematical Education in Science and Technology, DOI: 10.1080/0020739X.2016.1210245

To link to this article: http://dx.doi.org/10.1080/0020739X.2016.1210245

Published online: 03 Aug 2016.

INTERNATIONAL JOURNAL OF MATHEMATICAL EDUCATION IN SCIENCE AND TECHNOLOGY, 2016
http://dx.doi.org/10.1080/0020739X.2016.1210245

CLASSROOM NOTE

An explanatory, transformation geometry proof of a classic treasure-hunt problem and its generalization

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ABSTRACT

This paper discusses an interesting, classic problem that provides a nice classroom investigation for dynamic geometry, and which can easily be explained (proved) with transformation geometry. The deductive explanation (proof) provides insight into why it is true, leading to an immediate generalization, thus illustrating the discovery function of logical reasoning (proof).

ARTICLE HISTORY
Received 12 May 2016

KEYWORDS
Treasure-hunt problem; pirate problem; dynamic geometry; rotations; composition of rotations; dilations; spiral similarity

SUBJECT CLASSIFICATION CODES
97G50; 97D40; 97D50

1. Introduction

Consider[1] the following interesting treasure-hunt problem apparently first given by Gamow,[1] though also again discussed by Srinivasan [2]:

> 'A man finds an old document left by his grandfather. The document gives specific directions for reaching a certain island. On this island, he has to locate a specific oak tree, a specific pine tree and a gallows. He is asked to start from the gallows and march up to the oak tree counting the number of steps needed to reach the oak tree. At the oak tree, he must turn to the right and march the same number of steps. At this spot he must put a spike in the ground. He must then return to the gallows and then march to the pine tree counting the number of steps needed to reach the pine tree. At the pine tree, he must now turn to the left and march the same number of steps. At the second spot he has to put another spike in the ground. He must then dig at the midpoint of the line segment connecting the two spikes to recover the treasure buried there. Upon reaching the island, the man easily finds the specific oak tree and the specific pine tree. However, much to his dismay, he cannot locate the gallows. Rain and decay completely obliterated any traces of the place where the gallows once stood. Can the man still find the treasure? If so, how?'

To better appreciate the problem, Contreras [3] suggested letting students use interactive geometry software to explore this problem. The reader is now also encouraged before reading further to first dynamically explore the problem with the Java applet at: http://dynamicmathematicslearning.com/treasure-hunt.html.

Most surprisingly, the position of the treasure is completely independent of the position of the gallows and entirely determined by the positions of the two trees! In other words, the man could start from any arbitrary position for the gallows and still locate the treasure.

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How is this possible? It seems quite impossible and counter-intuitive. Perhaps some readers may even doubt this seemingly preposterous claim. If so, readers are strongly encouraged at this point to again test the claim by using the above link or making their own appropriate constructions in a dynamic geometry environment such as *Sketchpad*, *Cabri* or *GeoGebra*. Though such experimental confirmation is likely to strongly convince the reader, it is hoped that the reader now also has a burning desire to know how this surprising result can be explained.

2. Explanatory proofs

In agreement with Hanna,[4] an explanatory proof can be defined as a proof that not only verifies the mathematical correctness of a statement, but also additionally, provides deeper insight into why it is true. Not only do mathematicians generally prefer such 'illuminating' proofs (e.g. [5,6]), from a pedagogical point of view such proofs are especially desirable within a dynamic geometry context where students already have obtained high levels of conviction through continuous manipulation of a sketch and the testing of numerous cases. Moreover, an explanatory proof, since it reveals the underlying properties and reasons for the validity of a result, may upon further reflection, sometimes lead to the discovery of immediate generalizations, specializations or other variations. Such a process is illustrative of the so-called 'discovery function' of proof.[7]

From personal experience, the treasure-hunt problem stated above is a good example of a problem that when used with students or teachers creates surprise and astonishment,[8] and therefore naturally arouses their curiosity about why it is true. Due to the convincing nature of a dynamic geometry context, students are further likely to express a greater need for explanation than for conviction,[9,10] and an explanatory proof using transformation geometry as will be illustrated below, has the added benefit of automatically leading to some further generalizations.

3. Formulation of problem

Mathematically, the problem can be formulated as follows: let A and B be two distinct points in the plane (see Figure 1). Start from an arbitrary point P in the plane. After reaching the point A from P, turn to the right at a right angle and mark of the point S_1 such that $PA = AS_1$. Proceed from P to point B and turn to the left at right angles and mark off the point S_2 such that $PB = BS_2$. Show that the midpoint T of segment S_1S_2 is independent of the position of P.

Though it is possible to use coordinate geometry to show that the midpoint T (i.e. the treasure) is independent from the position of P (i.e. the gallows), this is likely to involve some tedious calculation. Alternatively, complex numbers could be used as illustrated by [1] or,[2] or one could consider vectors. Personally such methods, although powerful, are somewhat algorithmic, and sometimes not very explanatory of why a result is true. As already mentioned, the purpose of this paper is to provide a purely geometric proof that not only provides greater insight into why the result is true, but also leads to some generalizations.

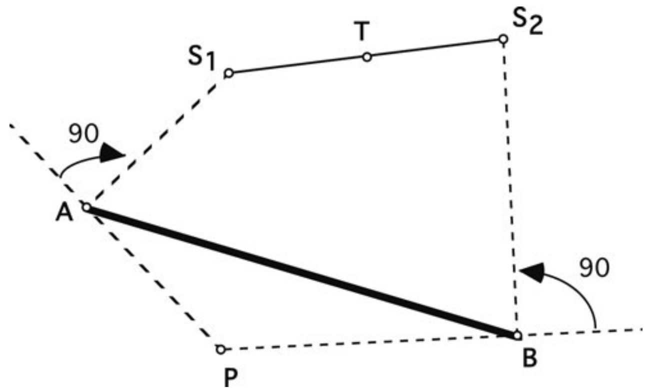

Figure 1. Formulation of problem.

4. Lemma

The treasure-hunt result follows directly from the following very useful theorem from transformation geometry, which deserves to be better known: 'The sum of two rotations with centres A and B through angles a and b, respectively, is a rotation through the angle $a + b$ around some centre O.'

4.1. Proof

It is insightful to first consider a proof of this theorem. Consider Figure 2. The sum of the two rotations carries the centre A of the first into a point A' such that $A'B = AB$ and angle $ABA' = b$. (The first rotation leaves A in place and the second carries A into A'). The sum of the two rotations also carries a point B' into B such that $B'A = BA$ and angle $BAB' = a$. (The first rotation carries B' into B and the second leaves B in place).

From this it follows that the centre O we are seeking is equidistant from B and B' and from A' and A; consequently, it can be found as the point of intersection of the perpendicular

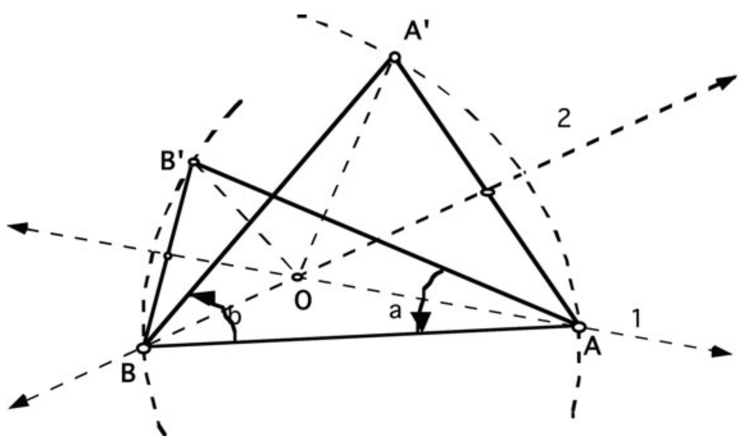

Figure 2. Sum of two rotations.

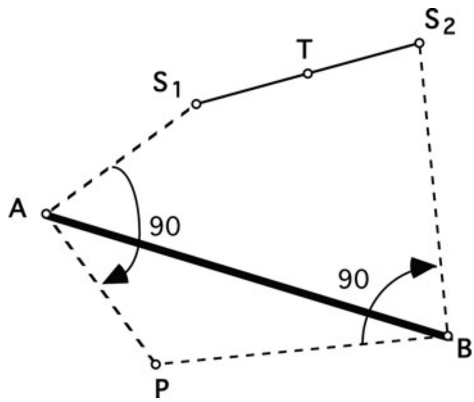

Figure 3. Proof of treasure-hunt problem.

bisectors l_1 and l_2 of the segments BB' and $A'A$, respectively. But from the figure, it is clear that l_1 passes through A and angle $OAB = a/2$, and that l_2 passes through B and angle $ABO = b/2$. The lines l_1 and l_2 are completely determined by these conditions; therefore, we have found the desired centre of rotation O. Finally, it should now be clear that angle $BOB' = a + b =$ angle AOA' so that a rotation of $a + b$ around O maps B' onto B and A onto A'.

5. Proof of treasure-hunt problem

Let us now return to our original problem by considering Figure 3. As shown, the original configuration is equivalent to two clockwise rotations, namely, a clockwise rotation of 90° of S_1 around A followed by another clockwise rotation of 90° of P around B. Therefore, S_2 is the image of S_1 by the two rotations, respectively, around A and B. But as we have seen from the Lemma above, the sum of these two rotations is equivalent to a half-turn (180°), whose centre must, therefore, be located at the midpoint of S_1S_2. This centre of the half-turn is completely determined as shown above by points A and B, and the sum of the two rotations being equal to a half-turn.

From the above Lemma, the position of T can be obtained directly from the configuration shown in Figure 4 (an adaptation of Figure 2 for this particular case). It follows easily that $B'ABA'$ is a square with T located at the intersection of its diagonals. This completes the proof that T is fixed, and therefore that its position is independent of P.

If X is, therefore, the midpoint of AB, T can also be located simply by constructing segment XT perpendicular to AB (in the appropriate direction), and equal to $AB/2$. On the other hand, since T is fixed as shown, to locate the treasure buried at T, the man may simply choose any point P as the gallows to start from, and carry out the prescribed procedure.

6. Looking back and generalizing

Looking back in the style of Polya [11] on the Lemma and the proof of the treasure-hunt problem, it can be observed that the result holds if the sum of the rotations is a half-turn. From this observation it follows immediately that T would be independent of P for any two angles of rotation, provided they sum to 180°. In fact, the result can be generalized to three

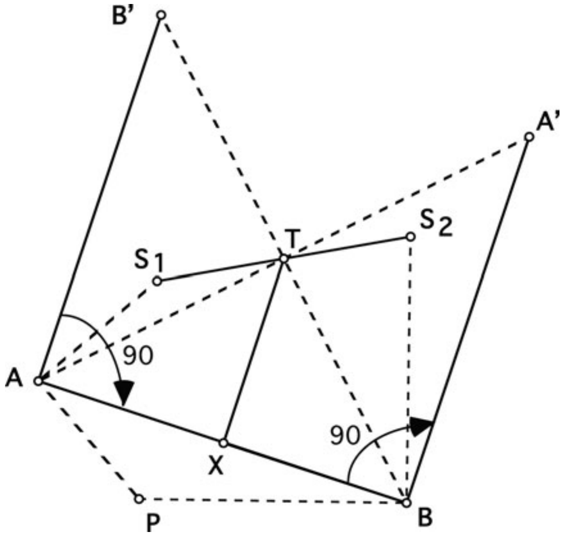

Figure 4. Location of treasure.

(or more) rotations at points A, B and C as shown in Figure 5, provided the angles a, b and c sum to 180 °. (In terms of the formulation of the original problem, one could start from any arbitrary point P, turn right through 180 °– a at A, left at B through 180 °–b and left at C through 180 °–c.)

The production of the transformation geometry proof using the Lemma, therefore, leads to two immediate generalizations by providing critical insight into why the result is true (compare with the discovery function of proof in [7]).

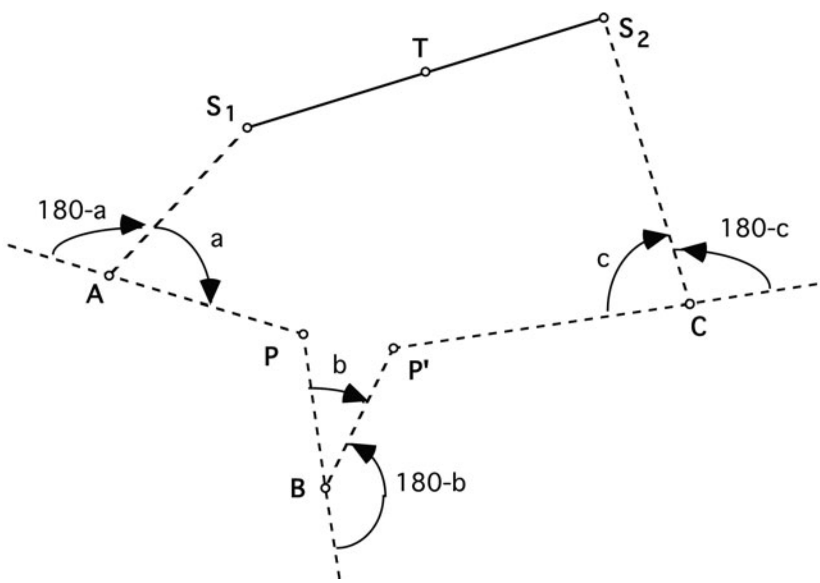

Figure 5. Looking back and generalizing.

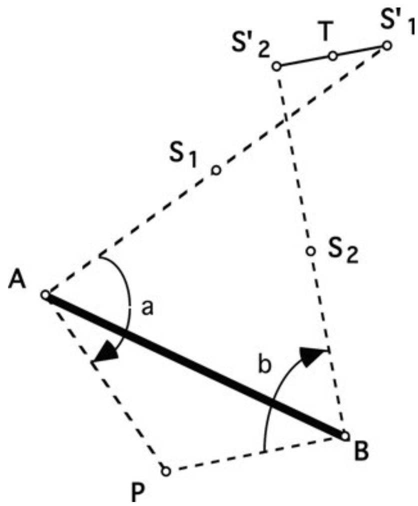

Figure 6. Further generalization with a spiral similarity.

7. Further generalization

The result that T is independent of P can even be further generalized by using the idea of a spiral similarity, denoted by (k, z) and which is defined as the sum (composition) of a dilation with factor k and a rotation of angle z about a fixed centre. Now consider the following result, which is merely a special case of a more general theorem given in Yaglom [12]: 'The sum of two spiral similarities (k, a) and $(1/k, b)$ around centres A and B is a half-turn, if $a + b = 180$ degrees.'

Using spiral similarities, we could, therefore, further generalize the result so that $AS'_1 = AP/k$ and $BS'_2 = BP/k$. An example is shown in Figure 6 where the dilation factor $k = 1/2$.

Finally, it is left as a challenge to the reader to use coordinate geometry or complex numbers to produce alternative proofs of these two generalizations.

8. Another application

Flores [13] gives the following result and then ingeniously uses velocity arguments to prove it:

Let ABC be any triangle, and choose any point D. Points D_1, D_2 and D_3 are obtained in the following way: D_1 is the reflection of D around B, D_2 is the reflection of D_1 around A and D_3 is the reflection of D_2 around C. Then the position of the midpoint M between D and D_3 is fixed.

However, since this construction is equivalent to three consecutive half-turns of D, respectively around B, A and C, and hence equivalent to one half-turn in total, the result immediately follows from the first generalization given above.

9. Concluding remarks

Generally there is an educational need to design learning activities for students at school and university to introduce them to both the explanatory power of proof, as well as the discovery function of proof. Using the example discussed here or one similar to it, could at least acquaint students with the idea that a deductive argument can provide additional insight, and sometimes produce novel, unexpected discoveries (compare Leong et al. [14]).

Note

1. This paper is based on talks given at the Annual Congress of the Association of Mathematics Education of South Africa (AMESA), 3–7 July 2000, University of Free State, Bloemfontein, South Africa, as well as at the Annual Congress of the Mathematical Association (MA), 13–16 April 2004, University of York, United Kingdom.

Disclosure statement

No potential conflict of interest was reported by the author.

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