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A trisection concurrency: a variation on a median theme

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"*We have to reinvent the wheel every once in a while, not because we need a lot of wheels: but we need a lot of inventors*." - Bruce Joyce in Serra (2008, p. 96)

Introduction

Euclidean geometry is a rich area in which to provide learners with some insight and opportunity to engage in the process of mathematical (re)discovery and (re)invention. According to Serra (2008, p. 96), the study of geometry in Egypt and Babylonia was based on observation and measurement over a period of time and was rooted largely in inductive reasoning in the early beginning. In fact, most scientific inquiry begins with inductive reasoning, usually leading to some unproved generalization (conjecture). It is therefore of general importance to engage learners in some inductive reasoning in geometry using construction, measurement and observation, both by hand and through the use of computing technology.

In South Africa, geometry has been challenging to learners and educators and proofs especially have been a major problem for learners as well as teachers. The re-introduction of geometry into the Curriculum Assessments Policy Statements (CAPS) to improve mathematics achievements of learners at the end of the Further Education and Training (FET) band (DoE, 2012) poses several educational challenges. However, university lecturers in mathematics, science and engineering believe that geometry is an integral part of mathematics, and teaching it at school is essential for university preparation. Several workshops to prepare teachers to teach geometry in the FET band have been held in this regard at the Edgewood and Westville campuses at the University of KwaZulu-Natal (UKZN).

A focus of these workshops has also been to not only concentrate on the verification function of proof, but also other functions of proof such as explanation, discovery, systematization, communication, and intellectual challenge (De Villiers, 1997). Using stimulating presentations of geometric results as suggested by Movshovitz-Hadar (1988) to solicit the surprise and curiousity of learners, one can raise their needs for explanation and

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intellectual challenge. Through scaffolded guidance, learners can than be led to rediscover and reinvent proofs that help them not only understand why the result is true, but also so that they feel empowered.

Concurrency

Apart from the important concepts of parallelness, similarity and congruency, concurrency is an extremely important concept in geometry. In general, three or more lines (or curves) in a plane are said to be concurrent if they intersect at a single point. Even though FET mathematics learners are unfortunately no longer required to know the proofs for the matric examination, they should at least be made familiar with the following famous points of concurrency in a triangle (see Figure 1):

i) The medians of a triangle are concurrent at the centroid (which divides the medians into the ratio 2:1, and is the centre of gravity of the triangle).

ii) The perpendicular bisectors of the sides of a triangle are concurrent at the circumcentre of the triangle. (The circumcentre is equidistant from the vertices).

iii) The angle bisectors of a triangle are concurrent at the incentre of the triangle. (The incentre is equidistant from the sides).

iv) The altitudes of a triangle are concurrent at the orthocenter.

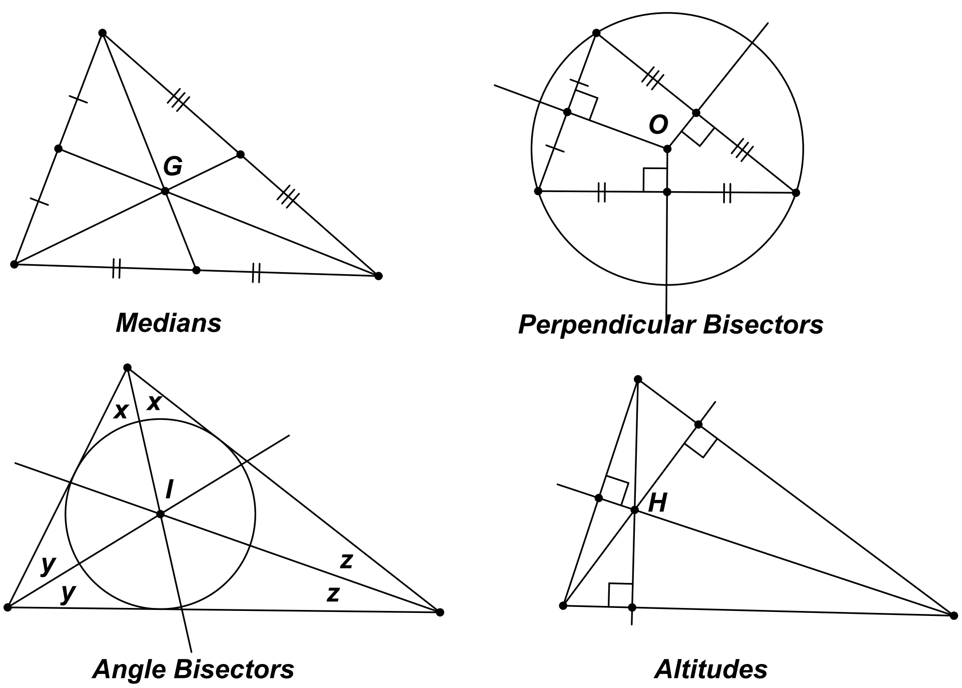

Figure 1: Some concurrencies in a triangle

Of interest is that none other than Albert Einstein remarked in his autobiography as follows about the indelible impression the concurrency of the altitudes, and its proof, made on his young mind (see Pyenson, 1985): "*At the age of 12, I experienced a second wonder of a totally different nature: in a little book dealing with Euclidean plane geometry, which came into my hands at the beginning of a school year. Here were assertions, as for example the intersection of the three altitudes of a triangle in one point, which — though by no means evident — could nevertheless be proved with such certainty that any doubt appeared to be out of the question. This lucidity and certainty made an indescribable impression upon me*."

Inductive, experimental discovery

Recently one of us (SP) made the following interesting experimental discovery using *Sketchpad*. It could be a challenging result not only for preservice and inservice mathematics teachers, but also for more mathematically talented learners who participate in mathematics olympiads. Other dynamic geometry software such as *GeoGebra*, *Cinderella*, etc. could also be used. The result is also instructive since it lends itself to some different approaches and proofs.

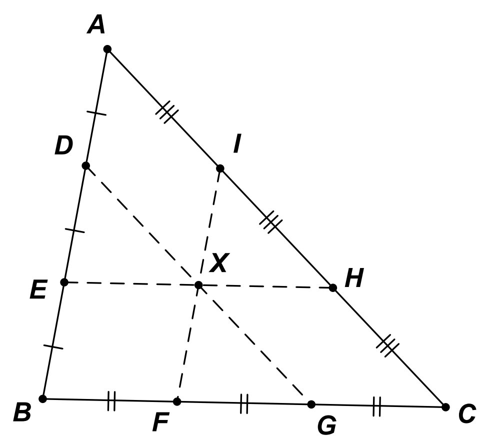

Figure 2: Trisection concurrency conjecture

Conjecture

Trisect the sides of a triangle *ABC* as shown in Figure 2. Then *DG*, *EH* and *FI* are concurrent in *X*.

Before continuing, the reader is invited to first experimentally explore and to become convinced of the conjecture above by using the interactive Java sketch at: http://dynamicmathematicslearning.com/trisection-concurrency.html

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Such inductive, empirical exploration by dragging the configuration into various positions is highly convincing, and frequently a *prerequisite* for starting to look for a proof, not as a means of verification/conviction, but as a means of explanation (compare De Villiers, 1997). The question is not WHETHER the experimentally discovered and confirmed result is true, but rather understanding WHY it is true.

Proof as a means of explanation

To explain why (prove that) this result is true, it is useful to first prove the following lemma.

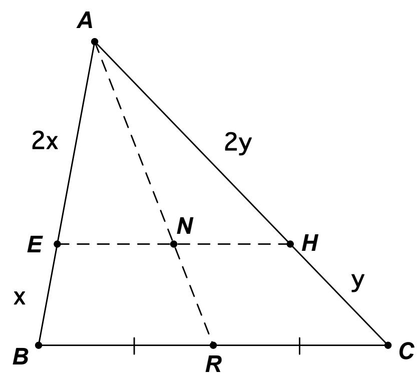

Figure 3

Lemma

Given ΔABC with $AE:EB = 2:1$ and $AH:HC = 2:1$. Then the midpoint N of EH is the centroid of ΔABC (see Figure 3).

Construction: Draw median AR with R on BC to intersect EH at N.

Proof

Since $\frac{AE}{EB} = \frac{AH}{HC} = \frac{2}{1}$, we have $EH \parallel BC$. This implies that $\frac{AE}{EB} = \frac{AN}{NR} = \frac{2}{1}$.

But AR is the median, so point N is the centroid of ΔABC. Triangles AEN and ABR are similar; hence $\frac{EN}{BR} = \frac{AN}{AR} \Rightarrow EN = BR.\frac{AN}{AR}$. Since triangles ANH and ARC are also similar, it follows in the same way that $NH = RC.\frac{AN}{AR}$. But $BR = RC$; thus $EN = NH$.

Proof of the conjecture

From the lemma above with reference to Figure 2, it now easily follows that the midpoints of EH, FI and DG are the centroid of the triangle, but since the centroid is a unique point in the triangle, these lines are concurrent at the centroid.

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An alternative proof as a means of discovery

As pointed out be De Villiers (1997), a proof of a result often leads to new insights and hence to the discovery of new or additional properties. The following approach uncovers more interesting properties, which also leads naturally to further generalization.

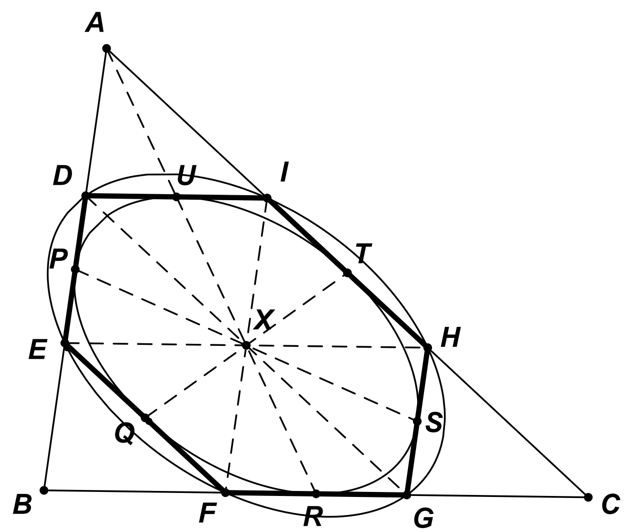

Figure 4: An alternative proof leading to additional properties

Proof

Since ∠*A* is common and the sides surrounding *A* are in the same ratio, it follows that Δ*ADI* /// Δ*AEH* /// Δ*ABC*. Hence, *DI* // *BC* since corresponding ∠*D* = ∠*B*. More over, since the scale factor between Δ*ADI* and Δ*ABC* is 3, it follows that *DI* //= *FG*. In the same way it can be shown that *DE* //= *HG* and *EF* //= *IH*, and therefore that *DEFGHI* is a *parallelo-hexagon*; in other words a hexagon with opposite sides parallel and equal. It is well known (and easy to prove) that just like a parallelogram, a parallelo-hexagon has half-turn symmetry and that its diagonals are concurrent at its centre of rotational symmetry. Thus *DG*, *EH* and *FI* are concurrent at *X*.

From the rotational symmetry it also follows that the lines connecting the midpoints of opposite sides, *PS*, *QT* and *RU*, are also concurrent at *X*. Furthermore, since *A* is the common centre of similarity of Δ*ADI*, Δ*AEH* and Δ*ABC*, it follows that the dilation of *R* (the midpoint of *BC*) respectively by (1/3) and (2/3) will have as respective images the midpoints of *DI* and *EH*, namely, *U* and *X*. It therefore follows that *X* is the centroid of the triangle.

Deducing more properties

Since opposite sides of the hexagon *DEFGHI* are parallel and thus meet at points at infinity, and are therefore collinear on the line of infinity, it follows from the converse of Pascal's theorem[1], that *EFGHI* lies on a conic (an ellipse in this case). In addition, since the diagonals of *DEFGHI* are concurrent, it follows from the converse of Brianchon's theorem[2] that *DEFGHI* also has an inscribed conic (an ellipse), which as a consequence of its rotational symmetry touches the parallelo-hexagon at *P*, *Q*, *R*, *S*, *T* and *U*, and has the same centre *X* as the circumscribed ellipse.

The ratio of the area of the triangle to that of the hexagon is also constant and can be easily deduced as follows. The area Δ*ADI* = 1/9 area Δ*ABC* since the scale factor between the two triangles is 1/3. Since triangles *ADI*, *EBF* and *HGC* have equal areas, it therefore follows that $\frac{Area\ \Delta ABC}{Area\ DEFGHI} = \frac{Area\ \Delta ABC}{\frac{2}{3}Area\ \Delta ABC} = \frac{3}{2}$.

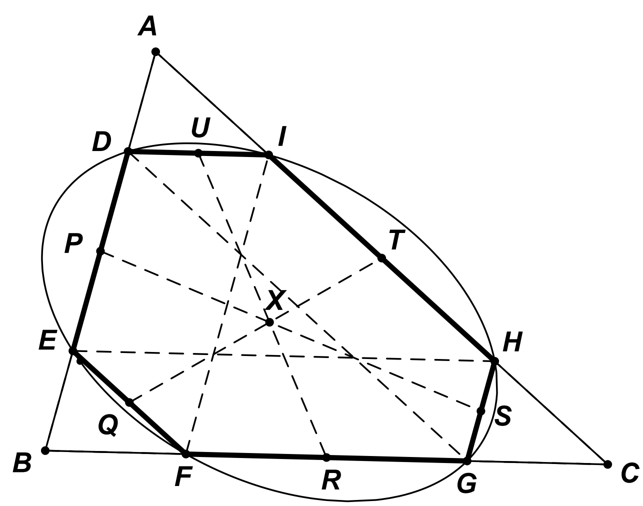

Figure 5: Generalizing to dividing the sides into 4 parts

Further generalization

If the sides are divided into 4 or more equal parts, the result generalizes to a hexagon with opposite sides parallel (but no longer equal), still inscribed in an ellipse, but its diagonals *DG*, *EH* and *FI* are no longer concurrent (see Figure 5). However, the lines connecting the midpoints of the parallel opposite sides are still concurrent[3] at *X*, which remains the centroid of Δ*ABC*. This

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[1] Read about Pascal's theorem at: http://en.wikipedia.org/wiki/Pascal's_theorem

[2] Read about Brianchon's theorem at: http://en.wikipedia.org/wiki/Brianchon%27s_theorem

[3] In general, from a result proved by De Villiers (2006) for any hexagon with opposite sides parallel, the lines connecting the midpoints of the opposite sides are concurrent. For a dynamic, interactive sketch go to: http://frink.machighway.com/~dynamicm/parahex.html

follows from the same pairwise similarity as before between each of the triangles *ADI*, *EBF*, *HGC* and ΔABC with the vertices *A*, *B*, and *C* respectively as the centres of similarity, which implies that these three lines coincide with the three medians.

In general, if the sides are divided into *n* equal parts ($n \geq 2$), then the area $\Delta ADI = 1/(n^2)$ area ΔABC.Thus, $\frac{Area\ \Delta ABC}{Area\ DEFGHI} = \frac{Area\ \Delta ABC}{\frac{n^2-3}{n^2} Area\ \Delta ABC} = \frac{n^2}{n^2-3}$. (Note when $n = 2$, the hexagon becomes a triangle).

Concluding remark

Though the results above are probably not new, they appear to not be well known. More importantly, they provide a nice challenge to extend high school learners and their teachers a little bit beyond the narrow confines of the curriculum.

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