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A note on special cases of Van Aubel's theorem

Research Article

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Abstract: In this short paper we study some properties of the lines associated with van Aubel's theorem in the special case when squares constructed on the sides of an arbitrary quadrilateral are replaced with equilateral triangles as well as isosceles triangles.

MSC: 97K30 • 68R10

Keywords: Van Aubel's theorem • Van Aubel's point • Orthodiagonal quadrilateral • Equilateral triangle • Kiepert hyperbola

1. Introduction (Van Aubel's theorem)

Consider an arbitrary Quadrilateral ABCD, the quadrilateral $S_1S_2S_3S_4$ formed by joining the four corresponding centers S_1, S_2, S_3, S_4 of the squares thus constructed on each side of ABCD is an iso-ortho diagonal quadrilateral [6]. That is $S_1S_3 = S_2S_4$ and $S_1S_3 \perp S_2S_4$. From Fig. 1 it is clear $S_1S_3 = S_2S_4$ and $S_1S_3 \perp S_2S_4$.

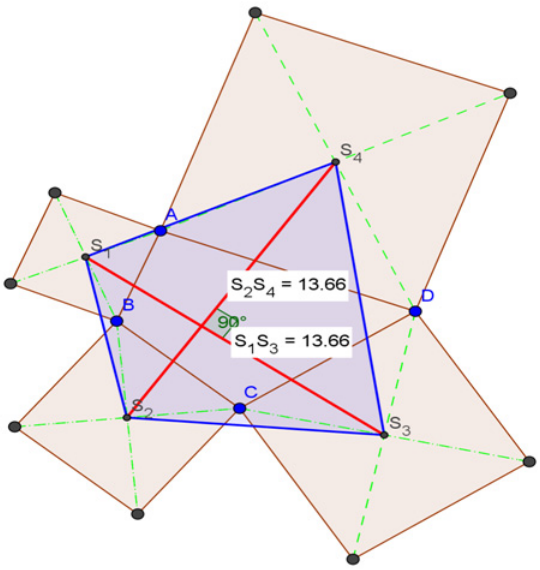

Fig. 1.

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The theorem we just stated above is attributed to Van Aubel (Von Aubel in [Gardner, p. 176-178]) could also be found in their work de Villiers, Yaglom, Finney among others.

In this article, we study the properties of the lines $S_1S_3$and $S_2S_4$when the squares are replaced with equilateral triangles and isosceles triangles, our present study about the special case of Van Aubel's theorem when the squares are replaced with equilateral triangles and further generalization is not actually new, since some of the authors studied about this earlier in 90's (can be found in [4, 5, 7–10]). Even though it is not a new study and the results presented in this article seems to be very elementary but are quite new and interesting. In this short note we also study about a point named as Van Aubel's point, its geometrical(ruler and compass) construction, its location in general case, and few more generalizations of van Aubel's theorem associated with Kiepert hyperbola.

2. Preliminaries

We use the following lemmas in proving the results.

Lemma 2.1.
If $A(x_1, y_1), B(x_2, y_2)$ are the two vertices of an arbitrary triangle ABC whose base angles are A and B then the coordinates of third vertex $C(x_3, y_3)$ is given by

$$\left(\frac{(x_1\tan A + x_2\tan B) \pm \tan A\tan B\,(y_1-y_2)}{\tan A+\tan B}\ ,\ \frac{(y_1\tan A + y_2\tan B) \mp \tan A\tan B\,(x_1-x_2)}{\tan A+\tan B}\right)$$

or

$$\left(\frac{(x_1\cot B + x_2\cot A) \pm (y_1-y_2)}{\cot A+\cot B}\ ,\ \frac{(y_1\cot B + y_2\cot A) \mp (x_1-x_2)}{\cot A+\cot B}\right)$$

Proof. Consider

$$\Lambda(\cot A+\cot B) = \det\begin{bmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ (x_1\cot B + x_2\cot A) \pm (y_1-y_2) & (y_1\cot B + y_2\cot A) \mp (x_1-x_2) & (\cot A+\cot B)\end{bmatrix}$$

By doing row operation on R_3 using R_1 and R_2, we get

$$\Lambda(\cot A+\cot B) = \begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ \pm(y_1-y_2) & \mp(x_1-x_2) & 0\end{vmatrix}$$

Which implies

$$\{\Lambda(\cot A+\cot B)\} = \pm\left[(x_1-x_2)^2+(y_1-y_2)^2\right] = \pm AB^2 \neq 0$$

We have area of triangle $ABC = \Delta$

$$= \frac{1}{2}\left|\det\left[\begin{matrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ \frac{(x_1\cot B+x_2\cot A)\pm(y_1-y_2)}{\cot A+\cot B} & \frac{(y_1\cot B+y_2\cot A)\mp(x_1-x_2)}{\cot A+\cot B} & 1\end{matrix}\right]\right|$$

$$= \frac{1}{2}\left|\frac{1}{\cot A+\cot B}\det\begin{bmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ (x_1\cot B + x_2\cot A) \pm (y_1-y_2) & (y_1\cot B + y_2\cot A) \mp (x_1-x_2) & \cot A+\cot B\end{bmatrix}\right|$$

$$= \frac{1}{2}\left|\frac{1}{\cot A+\cot B}\{\Lambda(\cot A+\cot B)\}\right|$$

$$= \frac{1}{2}\left|\frac{1}{\cot A+\cot B}(\pm AB^2)\right|$$

$$= \frac{AB^2}{2|\cot A+\cot B|} \neq 0 \quad (\text{since } \cot A+\cot B \neq 0)$$

It proves that area of triangle ABC is not equal to zero, which means that there is a triangle with A , B and C as vertices.

Now let us prove that base angles of triangle ABC are A and B, if the third vertex either C or C^1, where

$$C = \left(\frac{(x_1 \cot B + x_2 \cot A) + (y_1 - y_2)}{\cot A + \cot B}, \frac{(y_1 \cot B + y_2 \cot A) - (x_1 - x_2)}{\cot A + \cot B} \right)$$

and

$$C^1 = \left(\frac{(x_1 \cot B + x_2 \cot A) - (y_1 - y_2)}{\cot A + \cot B}, \frac{(y_1 \cot B + y_2 \cot A) + (x_1 - x_2)}{\cot A + \cot B} \right)$$

Clearly the midpoint D of C, C^1 lies on the line AB (since C^1 is the image of C with respect to the base AB of triangle ABC) its coordinate is given by,

$$D = \left(\frac{(x_1 \cot B + x_2 \cot A)}{(\cot A + \cot B)}, \frac{(y_1 \cot B + y_2 \cot A)}{(\cot A + \cot B)} \right)$$

And also D divides AB in the ratio given by

$$\frac{AD}{DB} = \frac{\cot A}{\cot B}$$

Hence

$$AD = \frac{AB \cot A}{\cot A + \cot B}, \; DB = \frac{AB \cot B}{\cot A + \cot B},$$

Now

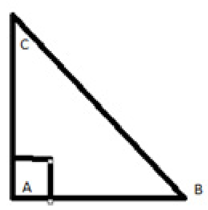
$$CD = CD' = \frac{2\Delta}{AB} = \frac{AB}{|\cot A + \cot B|} \quad \left(\text{Since } CD, CD^1 \text{ are the heights of the triangle } ABC, \text{ triangle } ABC^1\right)$$

hence

$$\frac{CD}{AD} = \frac{CD'}{AD} = \tan A, \frac{CD}{DB} = \frac{CD'}{DB} = \tan B$$

This proves that the base angles are A and B. □

Note: 2.1 is true even if one of the angles either A or B is right angle.



Proof. We start with $A = 90^o$, So, By considering the point C given in the 2.1, we have to prove that $CA$and AB are perpendicular to each other and $tanB = \frac{CA}{AB}$.
As we have angle $A = 90^0$, so $\cot A = 0$ and $\cot B \neq 0$. Hence

$$C = \left(\frac{(x_1 \cot B) + (y_1 - y_2)}{\cot B}, \frac{(y_1 \cot B) - (x_1 - x_2)}{\cot B} \right)$$

Slope of the line $CA = -\left(\frac{(x_1-x_2)}{(y_1-y_2)}\right)$, Slope of the line $AB = \left(\frac{(y_1-y_2)}{(x_1-x_2)}\right)$. It is clear that (slope of CA) (slope of AB)= -1, Hence $CA \perp AB$

$$\frac{CA}{AB} = \frac{1}{AB}\left(\sqrt{\left(\frac{y_1 - y_2}{\cot B}\right)^2 + \left(\frac{x_1 - x_2}{\cot B}\right)^2}\right) = \frac{1}{AB}\sqrt{\left(\frac{AB}{\cot B}\right)^2} = \tan B$$

This Proves that the point C so defined as in the statement of the lemma is, in fact, the third vertex of the triangle ABC, when $A = 90^0$. A nalogously, it is shown for C^1, the same occurs when $B = 90^0$. □

Corollary 2.1.
If $angleA = angleB = \theta$ that is triangle ABC is an isosceles triangle, then the coordinates of C are given by

$$\left(\frac{(x_1+x_2)\pm\tan\theta\left(y_1-y_2\right)}{2}\ ,\ \frac{\left(y_1+y_2\right)\mp\tan\theta\left(x_1-x_2\right)}{2}\right)$$

Corollary 2.2.
If $A = B = 60^o$ that is triangleABC is an equilateral triangle, then the coordinates of C are given by

$$\left(\frac{(x_1+x_2)\pm\sqrt{3}\left(y_1-y_2\right)}{2}\ ,\ \frac{\left(y_1+y_2\right)\mp\sqrt{3}\left(x_1-x_2\right)}{2}\right)$$

Lemma 2.2.
If $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ are the three vertices of an arbitrary triangle ABC then the coordinates of its circum center are given by

$$\left(\frac{x_1\sin2A+x_2\sin2B+x_3\sin2C}{\sin2A+\sin2B+\sin2C},\frac{y_1\sin2A+y_2\sin2B+y_3\sin2C}{\sin2A+\sin2B+\sin2C}\right)$$

where A, B, C are the angles of the triangle.

Corollary 2.3.
The coordinates of the circum center of an isosceles triangle whose vertices are $A(x_1, y_1), B(x_2, y_2)$ and $C\left(\frac{(x_1+x_2)\pm\tan\theta\left(y_1-y_2\right)}{2},\frac{\left(y_1+y_2\right)\mp\tan\theta\left(x_1-x_2\right)}{2}\right)$ Where θ is the base angle are given by $\left(\frac{(x_1+x_2)\mp\cot2\theta\left(y_1-y_2\right)}{2},\frac{\left(y_1+y_2\right)\pm\cot2\theta\left(x_1-x_2\right)}{2}\right)$.

Lemma 2.3.
If $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are the three vertices of an equilateral triangle then the coordinates of its center are given by $\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$.

Corollary 2.4.
The coordinates of the center of an equilateral triangle whose vertices are $(x_1, y_1), (x_2, y_2)$ and $\left(\frac{(x_1+x_2)\pm\sqrt{3}(y_1-y_2)}{2},\frac{(y_1+y_2)\mp\sqrt{3}(x_1-x_2)}{2}\right)$ are given by $\left(\frac{y_1-y_2+\sqrt{3}(x_1+x_2)}{2\sqrt{3}},\frac{x_2-x_1+\sqrt{3}(y_1+y_2)}{2\sqrt{3}}\right)$

3. Main results

Theorem 3.1.
If S_1, S_2, S_3S_4 are the centers of the equilateral triangles $\Delta ABP, \Delta BCQ, \Delta CDR, \Delta DAT$ are constructed which lie entirely out wards on the sides $AB = a, BC = b, CD = c$ and $AD = d$ of an arbitrary quadrilateral $ABCD$ respectively then the lines PR, QS are respectively perpendicular to the lines S_2S_4, S_1S_3 . That is $S_1S_3 \perp QT$, $S_2S_4 \perp PR$. [3]

Proof. With out loss of generality let us consider the coordinates of vertices of the quadrilateral $ABCD$ as $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$ and $D = (x_4, y_4)$. Then using 2.2, we have

$$P=\left(\frac{(x_1+x_2)+\sqrt{3}\left(y_1-y_2\right)}{2},\frac{\left(y_1+y_2\right)-\sqrt{3}\left(x_1-x_2\right)}{2}\right)$$

and

$$Q=\left(\frac{(x_2+x_3)+\sqrt{3}\left(y_2-y_3\right)}{2},\frac{\left(y_2+y_3\right)-\sqrt{3}\left(x_2-x_3\right)}{2}\right)$$

$$R=\left(\frac{(x_3+x_4)+\sqrt{3}\left(y_3-y_4\right)}{2},\frac{\left(y_3+y_4\right)-\sqrt{3}\left(x_3-x_4\right)}{2}\right)$$

$$T=\left(\frac{(x_4+x_1)+\sqrt{3}\left(y_4-y_1\right)}{2},\frac{\left(y_4+y_1\right)-\sqrt{3}\left(x_4-x_1\right)}{2}\right)$$

□

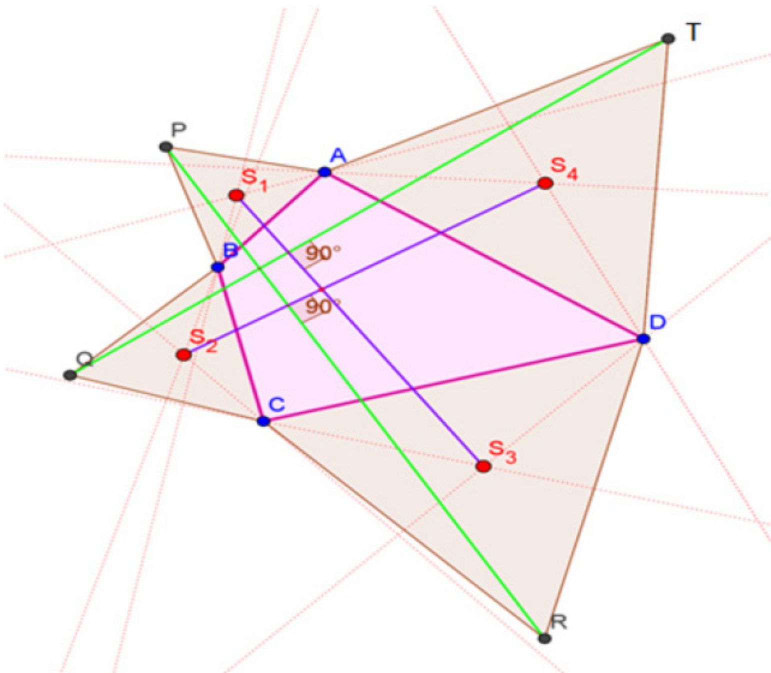


Fig. 2.

From Fig. 2, it is clear $S_1S_3 \perp QT$
and $S_2S_4 \perp PR$, and it is clear that

$$S_1 = \frac{A+B+P}{3} = \left(\frac{y_1 - y_2 + \sqrt{3}(x_1 + x_2)}{2\sqrt{3}}, \frac{x_2 - x_1 + \sqrt{3}(y_1 + y_2)}{2\sqrt{3}}\right)$$

$$S_2 = \frac{B+C+Q}{3} = \left(\frac{y_2 - y_3 + \sqrt{3}(x_2 + x_3)}{2\sqrt{3}}, \frac{x_3 - x_2 + \sqrt{3}(y_2 + y_3)}{2\sqrt{3}}\right)$$

$$S_3 = \frac{C+D+R}{3} = \left(\frac{y_3 - y_4 + \sqrt{3}(x_3 + x_4)}{2\sqrt{3}}, \frac{x_4 - x_3 + \sqrt{3}(y_3 + y_4)}{2\sqrt{3}}\right)$$

$$S_4 = \frac{D+A+T}{3} = \left(\frac{y_4 - y_1 + \sqrt{3}(x_4 + x_1)}{2\sqrt{3}}, \frac{x_1 - x_4 + \sqrt{3}(y_4 + y_1)}{2\sqrt{3}}\right)$$

So,

$$\text{Slope of the line } PR = \left(\frac{(y_1 + y_2 - y_3 - y_4) + \sqrt{3}(x_2 + x_3 - x_4 - x_1)}{(x_1 + x_2 - x_3 - x_4) - \sqrt{3}(y_2 + y_3 - y_4 - y_1)}\right)$$

$$\text{Slope of the line } QT = \left(\frac{(y_2 + y_3 - y_4 - y_1) + \sqrt{3}(x_3 + x_4 - x_1 - x_2)}{(x_2 + x_3 - x_4 - x_1) - \sqrt{3}(y_3 + y_4 - y_1 - y_2)}\right)$$

$$\text{Slope of the line } S_2S_4 = -\left(\frac{(x_1 + x_2 - x_3 - x_4) - \sqrt{3}(y_2 + y_3 - y_4 - y_1)}{(y_1 + y_2 - y_3 - y_4) + \sqrt{3}(x_2 + x_3 - x_4 - x_1)}\right)$$

$$\text{Slope of the line } S_1S_3 = -\left(\frac{(x_2 + x_3 - x_4 - x_1) - \sqrt{3}(y_3 + y_4 - y_1 - y_2)}{(y_2 + y_3 - y_4 - y_1) + \sqrt{3}(x_3 + x_4 - x_1 - x_2)}\right)$$

Now it is clear that (slope of PR) (slope of S_2S_4) $=-1$ = (slope of QT) (slope of S_1S_3).
Hence $S_1S_3 \perp QT$, $S_2S_4 \perp PR$

Theorem 3.2.
If S_1', S_2', S_3' and S_4' are the centers of the equilateral triangles $\Delta ABP'$, $\Delta BCQ'$, $\Delta CDR'$, $\Delta DAT'$ are constructed which lie entirely inwards on the sides $AB = a$, $BC = b$, $CD = c$ and $AD = d$ of an arbitrary quadrilateral ABCD respectively then the lines $P'R'$, $Q'T'$ are respectively perpendicular to the lines $S_2'S_4'$ and $S_1'S_3'$. That is $S_1'S_3' \perp Q'T'$, $S_2'S_4' \perp P'R'$.

Proof. Without loss of generality let us consider the coordinates of vertices of the quadrilateral $ABCD$ as $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$ and $D = (x_4, y_4)$. Then using 2.2, we have

$$P' = \left(\frac{(x_1+x_2)-\sqrt{3}\left(y_1-y_2\right)}{2}, \frac{\left(y_1+y_2\right)+\sqrt{3}\left(x_1-x_2\right)}{2}\right)$$

$$Q' = \left(\frac{(x_2+x_3)-\sqrt{3}\left(y_2-y_3\right)}{2}, \frac{\left(y_2+y_3\right)+\sqrt{3}\left(x_2-x_3\right)}{2}\right)$$

$$R' = \left(\frac{(x_3+x_4)-\sqrt{3}\left(y_3-y_4\right)}{2}, \frac{\left(y_3+y_4\right)+\sqrt{3}\left(x_3-x_4\right)}{2}\right)$$

$$T' = \left(\frac{(x_4+x_1)-\sqrt{3}\left(y_4-y_1\right)}{2}, \frac{\left(y_4+y_1\right)+\sqrt{3}\left(x_4-x_1\right)}{2}\right)$$

And it is clear that

$$S'_1 = \frac{B+C+Q'}{3} = \left(\frac{y_2-y_1+\sqrt{3}(x_1+x_2)}{2\sqrt{3}}, \frac{x_1-x_2+\sqrt{3}(y_1+y_2)}{2\sqrt{3}}\right),$$

$$S'_2 = \frac{C+D+R'}{3} = \left(\frac{y_3-y_2+\sqrt{3}(x_2+x_3)}{2\sqrt{3}}, \frac{x_2-x_3+\sqrt{3}(y_2+y_3)}{2\sqrt{3}}\right),$$

$$S'_3 = \frac{C+D+R'}{3} = \left(\frac{y_4-y_3+\sqrt{3}(x_3+x_4)}{2\sqrt{3}}, \frac{x_3-x_4+\sqrt{3}(y_3+y_4)}{2\sqrt{3}}\right),$$

$$S'_4 = \frac{D+A+T'}{3} = \left(\frac{y_1-y_4+\sqrt{3}(x_4+x_1)}{2\sqrt{3}}, \frac{x_4-x_1+\sqrt{3}(y_4+y_1)}{2\sqrt{3}}\right)$$

So,

$$\text{Slope of the line } P'R' = \left(\frac{\left(y_1+y_2-y_3-y_4\right)-\sqrt{3}(x_2+x_3-x_4-x_1)}{(x_1+x_2-x_3-x_4)+\sqrt{3}(y_2+y_3-y_4-y_1)}\right)$$

$$\text{Slope of the line } QT' = \left(\frac{\left(y_2+y_3-y_4-y_1\right)-\sqrt{3}(x_3+x_4-x_1-x_2)}{(x_2+x_3-x_4-x_1)+\sqrt{3}(y_3+y_4-y_1-y_2)}\right)$$

$$\text{Slope of the line } S'_2S'_4 = -\left(\frac{(x_1+x_2-x_3-x_4)+\sqrt{3}(y_2+y_3-y_4-y_1)}{\left(y_1+y_2-y_3-y_4\right)-\sqrt{3}(x_2+x_3-x_4-x_1)}\right)$$

$$\text{Slope of the line } S'_1S'_3 = -\left(\frac{(x_2+x_3-x_4-x_1)+\sqrt{3}(y_3+y_4-y_1-y_2)}{\left(y_2+y_3-y_4-y_1\right)-\sqrt{3}(x_3+x_4-x_1-x_2)}\right)$$

Now it is clear that (slope of $P'R'$) (slope of $S'_2S'_4$) =−1 = (slope of $Q'T'$) (slope of $S'_1S'_3$). Hence $S'_1S'_3 \perp Q'T', S'_2S'_4 \perp P'R'$. □

Theorem 3.3.
Let V_1, V_2, V_3 and V_4 are the points of intersection of the lines PR, QT, S_1S_3 and S_2S_4 then the four points V_1, V_2, V_3 and V_4 are concyclic (see Fig. 3).

Proof. From Theorem 3.1, it is clear that $V_1V_2 \perp V_2V_3$ and $V_3V_4 \perp V_4V_1$. Hence the four points V_1, V_2, V_3 and V_4 are concyclic which completes the proof of the Theorem 3.3. □

Theorem 3.4.
Let V'_1, V'_2, V'_3 and V'_4 are the points of intersection of the lines P'R', Q'T', $S'_1S'_3$ and $S'_2S'_4$ then the four points V'_1, V'_2, V'_3 and V'_4 are concyclic (see Fig. 4).

Proof. From Theorem 3.2, it is clear that $V'_1V'_2 \perp V'_2V'_3$ and $V'_3V'_4 \perp V'_4V'_1$. Hence the four points V'_1, V'_2, V'_3 and V'_4 are concyclic which completes the proof of the Theorem 3.4. □

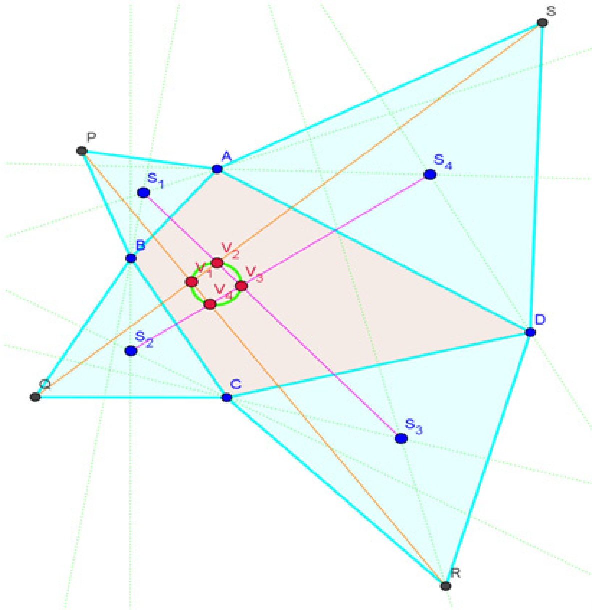


Fig. 3.

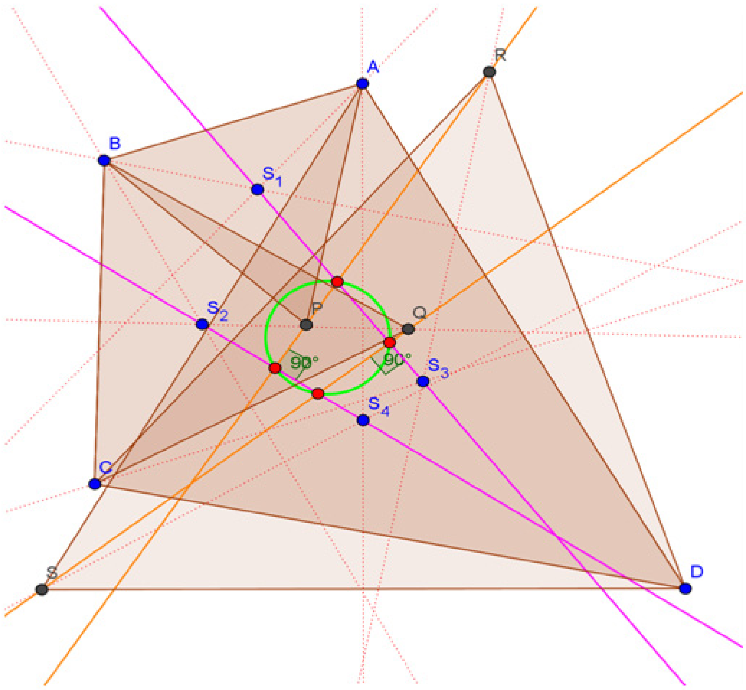


Fig. 4.

Theorem 3.5.
The quadrilaterals PQ'RT', P'QR'T, $S_1S_2'S_3S_4'$ and $S_1'S_2S_3'S_4$ are parallelograms.

Proof. To prove the quadrilateral $PQ'RT', P'QR'T, S_1S_2'S_3S_4', S_1'S_2S_3'S_4$ are parallelograms, It is enough to prove that diagonals bisect each other. It is clear that
The mid point of PR = The mid point of $Q'T'$ =

$$M_1 = \left(\frac{(x_1+x_2+x_3+x_4)+\sqrt{3}\left(y_1-y_2+y_3-y_4\right)}{4}, \frac{\left(y_1+y_2+y_3+y_4\right)-\sqrt{3}\left(x_1-x_2+x_3-x_4\right)}{4}\right)$$

The mid point of QT = The mid point of $P'R'$ =

$$M_2 = \left(\frac{(x_1+x_2+x_3+x_4) - \sqrt{3}\left(y_1-y_2+y_3-y_4\right)}{4}, \frac{\left(y_1+y_2+y_3+y_4\right) + \sqrt{3}(x_1-x_2+x_3-x_4)}{4} \right)$$

The mid point of S_1S_3 = The mid point of $S_2'S_4'$ =

$$M_3 = \left(\frac{\left(y_1-y_2+y_3-y_4\right) + \sqrt{3}(x_1+x_2+x_3+x_4)}{4\sqrt{3}}, \frac{-(x_1-x_2+x_3-x_4) + \sqrt{3}\left(y_1+y_2+y_3+y_4\right)}{4\sqrt{3}} \right)$$

The mid point of S_2S_4 = The mid point of $S_1'S_3'$ =

$$M_4 = \left(\frac{-\left(y_1-y_2+y_3-y_4\right) + \sqrt{3}(x_1+x_2+x_3+x_4)}{4\sqrt{3}}, \frac{(x_1-x_2+x_3-x_4) + \sqrt{3}\left(y_1+y_2+y_3+y_4\right)}{4\sqrt{3}} \right)$$

Hence, Theorem 3.5 is proved. □

Theorem 3.6.
Let M_1, M_2, M_3, M_4 are the point of intersections of the diagonals of the parallelograms PQ'RT', P'QR'T, $S_1S_2'S_3S_4'$ and $S_1'S_2S_3'S_4$ respectively then M_1, M_2, M_3, M_4 are collinear, and they lies on the line (for recognisation sake let us call this line as van aubel's line) given by

$$4(x_1-x_2+x_3-x_4)x + 4(y_1-y_2+y_3-y_4)y = (x_1+x_3)^2 - (x_2+x_4)^2 + \left(y_1+y_3\right)^2 - \left(y_2+y_4\right)^2$$

Proof. Consider $\lambda = x_1+x_2+x_3+x_4$, $\beta = y_1+y_2+y_3+y_4$, $\gamma = x_1-x_2+x_3-x_4$ and $\delta = y_1-y_2+y_3-y_4$. So

$$M_1 = \left(\frac{(x_1+x_2+x_3+x_4) + \sqrt{3}\left(y_1-y_2+y_3-y_4\right)}{4}, \frac{\left(y_1+y_2+y_3+y_4\right) - \sqrt{3}(x_1-x_2+x_3-x_4)}{4} \right) = \left(\frac{\lambda+\sqrt{3}\delta}{4}, \frac{\beta-\sqrt{3}\gamma}{4} \right)$$

$$M_2 = \left(\frac{(x_1+x_2+x_3+x_4) - \sqrt{3}\left(y_1-y_2+y_3-y_4\right)}{4}, \frac{\left(y_1+y_2+y_3+y_4\right) + \sqrt{3}(x_1-x_2+x_3-x_4)}{4} \right) = \left(\frac{\lambda-\sqrt{3}\delta}{4}, \frac{\beta+\sqrt{3}\gamma}{4} \right)$$

$$M_3 = \left(\frac{\left(y_1-y_2+y_3-y_4\right) + \sqrt{3}(x_1+x_2+x_3+x_4)}{4\sqrt{3}}, \frac{-(x_1-x_2+x_3-x_4) + \sqrt{3}\left(y_1+y_2+y_3+y_4\right)}{4\sqrt{3}} \right) = \left(\frac{\delta+\sqrt{3}\lambda}{4\sqrt{3}}, \frac{-\gamma+\sqrt{3}\beta}{4\sqrt{3}} \right)$$

and

$$M_4 = \left(\frac{-\left(y_1-y_2+y_3-y_4\right) + \sqrt{3}(x_1+x_2+x_3+x_4)}{4\sqrt{3}}, \frac{(x_1-x_2+x_3-x_4) + \sqrt{3}\left(y_1+y_2+y_3+y_4\right)}{4\sqrt{3}} \right) = \left(\frac{-\delta+\sqrt{3}\lambda}{4\sqrt{3}}, \frac{\gamma+\sqrt{3}\beta}{4\sqrt{3}} \right)$$

Consider a line

$$4\gamma x \ + \ 4\delta y \ = \ \lambda\gamma + \beta\delta \tag{1}$$

Clearly the four points M_1, M_2, M_3 and M_4 lies on this line (1). Hence the four points M_1, M_2, M_3 and M_4 are collinear. The line through these points is $4\gamma x \ + \ 4\delta y \ = \ \lambda\gamma + \beta\delta$. That is

$$4(x_1-x_2+x_3-x_4)x + 4(y_1-y_2+y_3-y_4)y = (x_1+x_3)^2 - (x_2+x_4)^2 + \left(y_1+y_3\right)^2 - \left(y_2+y_4\right)^2$$

□

Remark 3.1.

1. It is clear that the Mid Point of M_1M_2 = the Mid Point of $M_3M_4 = M = \left(\frac{\lambda}{4}, \frac{\beta}{4}\right) = \left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}\right)$.
2. The point M is also the point of intersection of diagonals of the parallelograms formed by joining the midpoints of the quadrilaterals $PQRS$ and $P'Q'R'S'$.
3. For recognization sake, let us call the point M as **Van Aubel's point** of the quadrilateral $ABCD$. (The point M acts as midpoint of the diagonals for any arbitrary parallelogram, rectangle, rhombus, square)

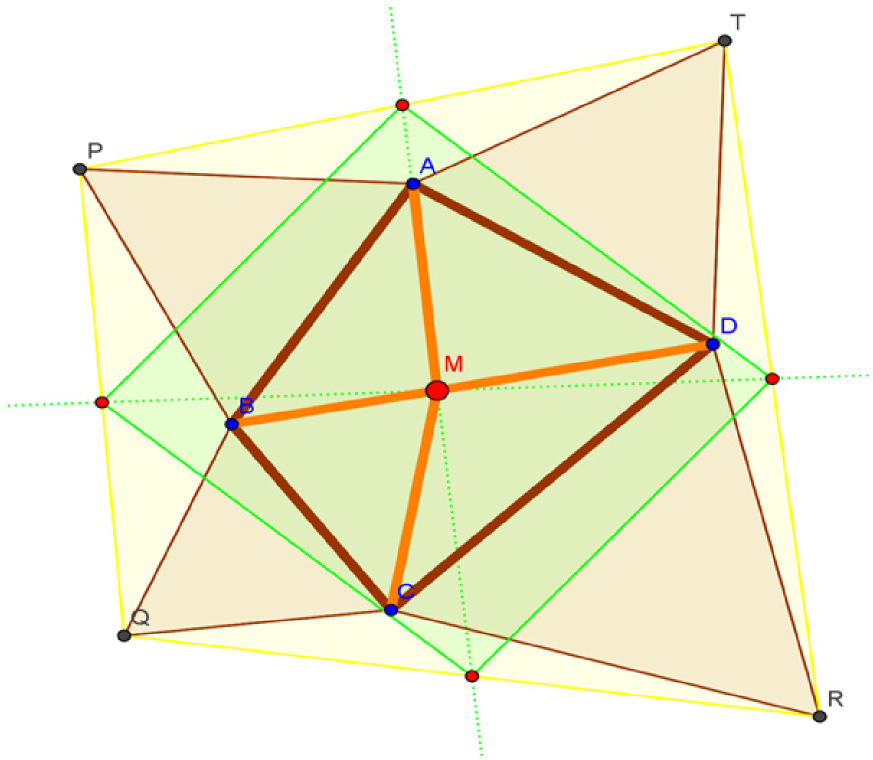


Fig. 5.

4. Using Theorem 3.5 and Theorem 3.6, it can also be stated as

   The midpoints of $PR\,(M_1)$, $S_1S_3\,(M_3)$, $QT\,(M_2)$, $S_2S_4\,(M_4)$ are collinear and M is the midpoint of M_1M_2 and M_3M_4.

   In the similar manner, the midpoints of $P'R'\,(M_2)$, $S_1'S_3'\,(M_4)$, $Q'T'\,(M_1)$, $S_2'S_4'\,(M_3)$ are collinear and M is the midpoint of M_1M_2 and M_3M_4

5. Using Theorem 3.5 and Theorem 3.6, we can see how to locate the point M using only ruler and compass,

   If some arbitrary quadrilateral $ABCD$ is given, construct the equilateral triangles on the sides either inside or outside, Let P, Q, R, T be its affix vertices, locate the midpoints of the sides of quadrilateral $PQRT$, then the point of intersection of the diagonals of quadrilateral formed by the midpoints of sides of $PQRT$ is required M. (see Fig. 5)

6. If I_1, I_2, I_3, I_4 and O_1, O_2, O_3, O_4 and G_1, G_2, G_3, G_4 are incentres, circumcenters and centroids of the triangles ABM, BCM, CDM and DAM respectively then the sets $\{I_1, I_2, I_3, I_4\}$ and $\{O_1, O_2, O_3, O_4\}$and $\{G_1, G_2, G_3, G_4\}$ are con cyclic when $ABCD$ is kite or square. The orthocenters H_1, H_2, H_3, H_4 of the triangles ABM, BCM, CDM and DAM are collinear when $ABCD$ is kite and coincides with M when $ABCD$ is square (see Fig. 6).

3.1. Generalizations

Theorem 3.7.
If S_1, S_2, S_3, S_4 are the circumcenters of the isosceles triangles ΔABP, ΔBCQ, ΔCDR, ΔDAT whose base angle is θ constructed entirely out wards on the sides of quadrilateral ABCD Then

(a) The midpoints of $PR\,(M_1)$, $S_1S_3\,(M_3)$, $QT\,(M_2)$, $S_2S_4\,(M_4)$are collinear and lie on the van Aubel's line given by $4(x_1 - x_2 + x_3 - x_4)x + 4(y_1 - y_2 + y_3 - y_4)y = (x_1 + x_3)^2 - (x_2 + x_4)^2 + \left(y_1 + y_3\right)^2 - \left(y_2 + y_4\right)^2$

(b) Van Aubel's point (M) is the midpoint of M_1M_2 and M_3M_4 (see figure-7)

Proof. We have by 2.1, the coordinates of P, Q, R, S are given by

$$P = \left(\frac{(x_1 + x_2) + \tan\theta\left(y_1 - y_2\right)}{2}, \frac{\left(y_1 + y_2\right) - \tan\theta\,(x_1 - x_2)}{2}\right)$$

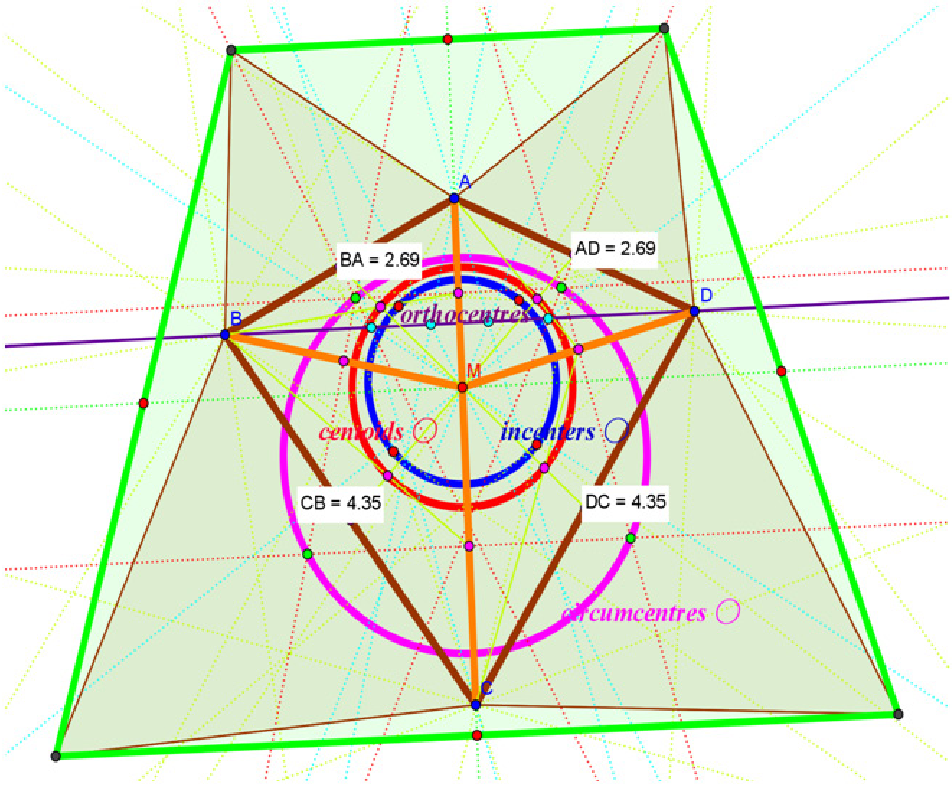


Fig. 6.

$$Q = \left(\frac{(x_2 + x_3) + \tan\theta\,(y_2 - y_3)}{2}, \frac{(y_2 + y_3) - \tan\theta\,(x_2 - x_3)}{2} \right)$$

$$R = \left(\frac{(x_3 + x_4) + \tan\theta\,(y_3 - y_4)}{2}, \frac{(y_3 + y_4) - \tan\theta\,(x_3 - x_4)}{2} \right)$$

$$T = \left(\frac{(x_4 + x_1) + \tan\theta\,(y_4 - y_1)}{2}, \frac{(y_4 + y_1) - \tan\theta\,(x_4 - x_1)}{2} \right)$$

And using 2.2, the circumcenters S_1, S_2, S_3 and S_4 are given by

$$S_1 = \left(\frac{(x_1 + x_2) - \cot 2\theta\,(y_1 - y_2)}{2}, \frac{(y_1 + y_2) + \cot 2\theta\,(x_1 - x_2)}{2} \right)$$

$$S_2 = \left(\frac{(x_2 + x_3) - \cot 2\theta\,(y_2 - y_3)}{2}, \frac{(y_2 + y_3) + \cot 2\theta\,(x_2 - x_3)}{2} \right)$$

$$S_3 = \left(\frac{(x_3 + x_4) - \cot 2\theta\,(y_3 - y_4)}{2}, \frac{(y_3 + y_4) + \cot 2\theta\,(x_3 - x_4)}{2} \right)$$

$$S_4 = \left(\frac{(x_4 + x_1) - \cot 2\theta\,(y_4 - y_1)}{2}, \frac{(y_4 + y_1) + \cot 2\theta\,(x_4 - x_1)}{2} \right)$$

The mid point of PR

$$M_1 = \left(\frac{(x_1 + x_2 + x_3 + x_4) + \tan\theta\,(y_1 - y_2 + y_3 - y_4)}{4}, \frac{(y_1 + y_2 + y_3 + y_4) - \tan\theta\,(x_1 - x_2 + x_3 - x_4)}{4} \right)$$

The mid point of QT

$$M_2 = \left(\frac{(x_1 + x_2 + x_3 + x_4) - \tan\theta\,(y_1 - y_2 + y_3 - y_4)}{4}, \frac{(y_1 + y_2 + y_3 + y_4) + \tan\theta\,(x_1 - x_2 + x_3 - x_4)}{4} \right)$$

The mid point of $S_1 S_3$

$$M_3 = \left(\frac{(x_1 + x_2 + x_3 + x_4) - \cot 2\theta\,(y_1 - y_2 + y_3 - y_4)}{4}, \frac{(y_1 + y_2 + y_3 + y_4) + \cot 2\theta\,(x_1 + x_2 + x_3 + x_4)}{4} \right)$$

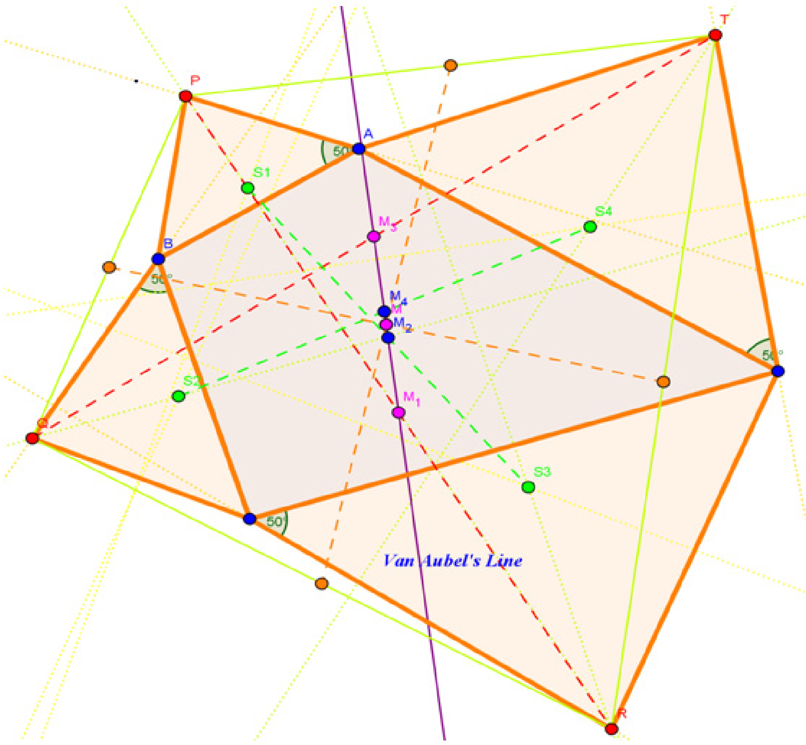


Fig. 7.

The mid point of S_2S_4

$$M_4 = \left(\frac{(x_1+x_2+x_3+x_4)+\cot 2\theta\,(y_1-y_2+y_3-y_4)}{4}, \frac{(y_1+y_2+y_3+y_4)-\cot 2\theta\,(x_1+x_2+x_3+x_4)}{4}\right)$$

Hence

$$M = \text{The Midpoint of } M_1'M_2' = \text{The Mid point of } M_3'M_4' = \left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}\right)$$

Hence (b) is proved.
Now to prove (a), consider $\lambda = x_1+x_2+x_3+x_4$, $\beta = y_1+y_2+y_3+y_4$, $\gamma = x_1-x_2+x_3-x_4$ and $\delta = y_1-y_2+y_3-y_4$, then

$$M_1 = \left(\frac{\lambda+\tan\theta\,\delta}{4}, \frac{\beta-\tan\theta\,\gamma}{4}\right)$$

$$M_2 = \left(\frac{\lambda-\tan\theta\,\delta}{4}, \frac{\beta+\tan\theta\,\gamma}{4}\right)$$

$$M_3 = \left(\frac{\lambda-\cot 2\theta\,\delta}{4}, \frac{\beta+\cot 2\theta\,\gamma}{4}\right)$$

and

$$M_4 = \left(\frac{\lambda+\cot 2\theta\,\delta}{4}, \frac{\beta-\cot 2\theta\,\gamma}{4}\right)$$

Consider a line

$$4\gamma x + 4\delta y = \lambda\gamma + \beta\delta \tag{2}$$

Clearly, the four points M_1, M_2, M_3 and M_4 lies on this line (2).
Hence, the four points M_1, M_2, M_3 and M_4 are collinear.
From Theorem 3.7, Clearly the line (2) is Van **Aubel's line (*L*)**. Its equation is given by

$$4(x_1-x_2+x_3-x_4)x + 4(y_1-y_2+y_3-y_4)y = (x_1+x_3)^2-(x_2+x_4)^2+(y_1+y_3)^2-(y_2+y_4)^2$$

□

Theorem 3.8.
If $S_1', S_2', S_3' S_4'$ are the circumcenters of an isosceles triangles $\Delta ABP', \Delta BCQ', \Delta CDR', \Delta DAT'$ whose base angle is θ', constructed entirely inwards on the sides of quadrilateral ABCD. Then

(a) *The midpoints of $P'R'\left(M_1'\right), S_1'S_3'\left(M_3'\right), Q'T'\left(M_2'\right), S_2'S_4'\left(M_4'\right)$ are collinear and lies on the Van Aubel's Line (**L**)*

(b) *Van Aubel's point (**M**) is the midpoint of $M_1'M_2'$ and $M_3'M_4'$*

Proof. We have by 2.1, the coordinates of P', Q', R', T' are given by

$$P' = \left(\frac{(x_1+x_2)-\tan\theta'\left(y_1-y_2\right)}{2}, \frac{\left(y_1+y_2\right)+\tan\theta'\left(x_1-x_2\right)}{2}\right)$$

$$Q' = \left(\frac{(x_2+x_3)-\tan\theta'\left(y_2-y_3\right)}{2}, \frac{\left(y_2+y_3\right)+\tan\theta'\left(x_2-x_3\right)}{2}\right)$$

$$R' = \left(\frac{(x_3+x_4)-\tan\theta'\left(y_3-y_4\right)}{2}, \frac{\left(y_3+y_4\right)+\tan\theta'\left(x_3-x_4\right)}{2}\right)$$

$$T' = \left(\frac{(x_4+x_1)-\tan\theta'\left(y_4-y_1\right)}{2}, \frac{\left(y_4+y_1\right)+\tan\theta'\left(x_4-x_1\right)}{2}\right)$$

And using 2.3, the circumcenters S_1', S_2', S_3' and S_4' are given by

$$S_1' = \left(\frac{(x_1+x_2)+\cot 2\theta'\left(y_1-y_2\right)}{2}, \frac{\left(y_1+y_2\right)-\cot 2\theta'\left(x_1-x_2\right)}{2}\right)$$

$$S_2' = \left(\frac{(x_2+x_3)+\cot 2\theta'\left(y_2-y_3\right)}{2}, \frac{\left(y_2+y_3\right)-\cot 2\theta'\left(x_2-x_3\right)}{2}\right)$$

$$S_3' = \left(\frac{(x_3+x_4)+\cot 2\theta'\left(y_3-y_4\right)}{2}, \frac{\left(y_3+y_4\right)-\cot 2\theta'\left(x_3-x_4\right)}{2}\right)$$

$$S_4' = \left(\frac{(x_4+x_1)+\cot 2\theta'\left(y_4-y_1\right)}{2}, \frac{\left(y_4+y_1\right)-\cot 2\theta'\left(x_4-x_1\right)}{2}\right)$$

The mid point of $P'R'$ =

$$M_1' = \left(\frac{(x_1+x_2+x_3+x_4)-\tan\theta'\left(y_1-y_2+y_3-y_4\right)}{4}, \frac{\left(y_1+y_2+y_3+y_4\right)+\tan\theta'\left(x_1-x_2+x_3-x_4\right)}{4}\right)$$

The mid point of $Q'T'$ =

$$M_2' = \left(\frac{(x_1+x_2+x_3+x_4)+\tan\theta'\left(y_1-y_2+y_3-y_4\right)}{4}, \frac{\left(y_1+y_2+y_3+y_4\right)-\tan\theta'\left(x_1-x_2+x_3-x_4\right)}{4}\right)$$

The mid point of $S_1'S_3'$ =

$$M_3' = \left(\frac{(x_1+x_2+x_3+x_4)+\cot 2\theta'\left(y_1-y_2+y_3-y_4\right)}{4}, \frac{\left(y_1+y_2+y_3+y_4\right)-\cot 2\theta'\left(x_1+x_2+x_3+x_4\right)}{4}\right)$$

The mid point of $S_2'S_4'$ =

$$M_4' = \left(\frac{(x_1+x_2+x_3+x_4)-\cot 2\theta'\left(y_1-y_2+y_3-y_4\right)}{4}, \frac{\left(y_1+y_2+y_3+y_4\right)+\cot 2\theta'\left(x_1+x_2+x_3+x_4\right)}{4}\right)$$

Hence M = The Midpoint of $M_1'M_2'$ = The Mid point of $M_3'M_4'$

$$= \left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}\right)$$

Hence (b) is proved.
Now to prove (a), Consider $\lambda = x_1+x_2+x_3+x_4$, $\beta = y_1+y_2+y_3+y_4$, $\gamma = x_1-x_2+x_3-x_4$ and $\delta = y_1-y_2+y_3-y_4$. Then

$$M_1' = \left(\frac{\lambda-\tan\theta'\delta}{4}, \frac{\beta+\tan\theta'\gamma}{4}\right)$$

$$M_2' = \left(\frac{\lambda+\tan\theta'\delta}{4}, \frac{\beta-\tan\theta'\gamma}{4}\right)$$

$$M'_3 = \left(\frac{\lambda + \cot 2\theta' \delta}{4}, \frac{\beta - \cot 2\theta' \gamma}{4}\right)$$

$$M'_4 = \left(\frac{\lambda - \cot 2\theta' \delta}{4}, \frac{\beta + \cot 2\theta' \gamma}{4}\right)$$

Consider a line

$$4\gamma x + 4\delta y = \lambda\gamma + \beta\delta \tag{3}$$

Clearly, the four points $M'_1, M'_2, M'_3$and M'_4 lies on this line (3). Hence, The four points $M'_1, M'_2, M'_3$and M'_4 are collinear.
The line through these points is $4\gamma x + 4\delta y = \lambda\gamma + \beta\delta$.
From Theorem 3.7, Clearly the line (3) is Van Aubel's line (L). □

Remark 3.2.

1. The Van Aubel's point (M) of the quadrilateral ABCD and the points M_1, M_2, M_3, M_4, M'_1, M'_2, M'_3 and M'_4 all lie on the Van Aubel's Line of the quadrilateral $ABCD$.
2. If θ and θ' of Theorem 3.7 and Theorem 3.8 are equal, Then the points M_1, M_2, M_3, M_4 respectively coincide with the points M'_1, M'_2, M'_3 and M'_4.

Theorem 3.9.
The quadrilaterals PQ'RT', P'QR'T, $S_1S'_2S_3S'_4$ and $S'_1S_2S'_3S_4$ are parallelograms where P, Q, R, T, P', Q', R', T' are the affixes of the isosceles triangles with base angles θ constructed on the sides of the quadrilateral ABCD out and inwards respectively.

Proof. To prove the quadrilateral $PQ'RT', P'QR'T, S_1S'_2S_3S'_4, S'_1S_2S'_3S_4$ are parallelograms, it is enough to prove that diagonals bisect each other.
By Theorem 3.7 and Theorem 3.8, it is clear that

The mid point of $PR =$ The mid point of $Q'T' =$

$$= \left(\frac{(x_1 + x_2 + x_3 + x_4) + \tan\theta\left(y_1 - y_2 + y_3 - y_4\right)}{4}, \frac{\left(y_1 + y_2 + y_3 + y_4\right) - \tan\theta\,(x_1 - x_2 + x_3 - x_4)}{4}\right)$$

The mid point of $QT =$ The mid point of $P'R' =$

$$= \left(\frac{(x_1 + x_2 + x_3 + x_4) - \tan\theta\left(y_1 - y_2 + y_3 - y_4\right)}{4}, \frac{\left(y_1 + y_2 + y_3 + y_4\right) + \tan\theta\,(x_1 - x_2 + x_3 - x_4)}{4}\right)$$

The mid point of $S_1S_3 =$ The mid point of $S'_2S'_4 =$

$$= \left(\frac{(x_1 + x_2 + x_3 + x_4) - \cot 2\theta\left(y_1 - y_2 + y_3 - y_4\right)}{4}, \frac{\left(y_1 + y_2 + y_3 + y_4\right) + \cot 2\theta\,(x_1 + x_2 + x_3 + x_4)}{4}\right)$$

The mid point of $S_2S_4 =$ The mid point of $S'_1S'_3 =$

$$= \left(\frac{(x_1 + x_2 + x_3 + x_4) + \cot 2\theta\left(y_1 - y_2 + y_3 - y_4\right)}{4}, \frac{\left(y_1 + y_2 + y_3 + y_4\right) - \cot 2\theta\,(x_1 + x_2 + x_3 + x_4)}{4}\right)$$

Hence theorem is proved. □

Theorem 3.10.
Suppose ABCD is a given arbitrary quadrilateral , let $P_1P_2...P_{2k+1}, Q_1Q_2...Q_{2k+1}, R_1R_2...R_{2k+1}, T_1T_2...T_{2k+1}$, $P'_1P'_2...P'_{2k+1}$, $Q'_1Q'_2...Q'_{2k+1}$, $R'_1R'_2...R'_{2k+1}$ and $T'_1T'_2...T'_{2k+1}$ be the regular polygons of $2k+1$ sides constructed on the sides of ABCD out and inwards respectively, where $k \geqslant 1$ such that $P_1P_{2k+1} = AB = P'_1P'_{2k+1}$, $Q_1Q_{2k+1} = BC = Q'_1Q'_{2k+1}$, $R_1R_{2k+1} = CD = R'_1R'_{2k+1}$, $T_1T_{2k+1} = DA = T'_1T'_{2k+1}$, and S_1, S_2, S_3, S_4, S'_1, S'_2, S'_3, S'_4 are the centers of the regular polygons constructed on the sides, then

(a) The midpoints of $P_{\frac{k+1}{2}}Q_{\frac{k+1}{2}}\,(M_1), S_1S_3\,(M_3), R_{\frac{k+1}{2}}T_{\frac{k+1}{2}}\,(M_2)$, $S_2S_4\,(M_4), P'_{\frac{k+1}{2}}Q'_{\frac{k+1}{2}}\left(M'_1\right), S'_1S'_3\left(M'_3\right)$, $R'_{\frac{k+1}{2}}T'_{\frac{k+1}{2}}\left(M'_2\right), S'_2S'_4\left(M'_4\right)$ are collinear and lie on the van Aubel's line(L) given by

$$4(x_1 - x_2 + x_3 - x_4)x + 4(y_1 - y_2 + y_3 - y_4)y = (x_1 + x_3)^2 - (x_2 + x_4)^2 + \left(y_1 + y_3\right)^2 - \left(y_2 + y_4\right)^2$$

(b) *The quadrilaterals* $P_{\frac{k+1}{2}} Q'_{\frac{k+1}{2}} R_{\frac{k+1}{2}} T'_{\frac{k+1}{2}}, P'_{\frac{k+1}{2}} Q_{\frac{k+1}{2}} R'_{\frac{k+1}{2}} T_{\frac{k+1}{2}}, S_1 S'_2 S_3 S'_4$ *and* $S'_1 S_2 S'_3 S_4$ *are parallelograms.*

Proof. It is clear that in the regular polygons $P_1P_2...P_{2k+1}, Q_1Q_2...Q_{2k+1}, R_1R_2...R_{2k+1}, T_1T_2...T_{2k+1}, P'_1P'_2...P'_{2k+1}, Q'_1Q'_2...Q'_{2k+1}, R'_1R'_2...R'_{2k+1}$ and $T'_1T'_2...T'_{2k+1}$, the triangles $AP_{\frac{K+1}{2}}B, BQ_{\frac{K+1}{2}}C, CR_{\frac{K+1}{2}}D, DT_{\frac{K+1}{2}}A$ are isosceles triangles with base angle θ constructed outwards on the sides AB, BC, CD, DA of quadrilateral $ABCD$, here as the triangles $AP'_{\frac{K+1}{2}}B, BQ'_{\frac{K+1}{2}}C, CR'_{\frac{K+1}{2}}D, DT'_{\frac{K+1}{2}}A$ are also isosceles triangles with base angle θ, constructed inwards on the sides AB, BC, CD, DA of quadrilateral $ABCD$.
Hence, By Theorem 3.7 and 3.8, (a) is true.
In the similar manner, we can prove (b) using Theorem 3.9. □

Remark 3.3.
Clearly, by Theorem 3.10, it is true that we can also plot **Van Aubel's Point (M)** for an arbitrary quadrilateral $ABCD$ by constructing the regular polygons of n number of sides on the sides of quadrilateral lie inwards or outwards, and by applying the procedure discussed in 3.1.

Theorem 3.11.
Let ABCD is a quadrilateral, Suppose the triangles $\Delta ABP, \Delta CDR$, *are isosceles with angle* α *at their top vertices, and* $\Delta BCQ, \Delta DAT$ *are isosceles with angle* $\pi - \alpha$ *at their top vertices (all of them have same orientation) constructed on the sides of the quadrilateral which lie outwards and if* S_1, S_2, S_3, S_4 *the circumcenters of the triangles* $\Delta ABP, \Delta BCQ, \Delta CDR$ *and* ΔDAT *then*

(a) *PR is perpendicular to QT.*

(b) *The ratio of these two segements, PR and QT doesn't depend from the quadrilateral.*

(c) *Quadrilateral* S_1, S_2, S_3, S_4 *is parallelogram.*

(d) *The three points, Van Aubel's Point* $(M_{S_1S_2S_3S_4})$ *of quadrilateral* $S_1S_2S_3S_4$ *and the mid points of* $PR(M_{PR})$ *and* $QT(M_{QT})$ *are collinear and lie on the line Van Aubel's Line given by*

$$4(x_1 - x_2 + x_3 - x_4)x + 4(y_1 - y_2 + y_3 - y_4)y = (x_1 + x_3)^2 - (x_2 + x_4)^2 + (y_1 + y_3)^2 - (y_2 + y_4)^2$$

And in particular the midpoint of $PR(MP_{R})$ *and* $QT(M_{QT})$ *is the Van Aubel's point of* $S_1S_2S_3S_4$ *.(see Fig. 8)*

Proof. Given at the top vertices P, R makes an angle Îś, so the two isosceles triangles $\Delta ABP, \Delta CDR$ having the base angle as $90^o - \alpha/2$. Using 2.1, we have

$$P = \left(\frac{(x_1 + x_2) + \cot\left(\frac{\alpha}{2}\right)(y_1 - y_2)}{2}, \frac{(y_1 + y_2) - \cot\left(\frac{\alpha}{2}\right)(x_1 - x_2)}{2} \right)$$

$$R = \left(\frac{(x_3 + x_4) + \cot\left(\frac{\alpha}{2}\right)(y_3 - y_4)}{2}, \frac{(y_3 + y_4) - \cot\left(\frac{\alpha}{2}\right)(x_3 - x_4)}{2} \right)$$

And given at the top vertices Q, T makes an angle $\pi - \alpha$, so the two isosceles triangles $\Delta BCQ, \Delta DAT$ having the base angle as $\alpha/2$. Hence, using 2.2, we have

$$Q = \left(\frac{(x_2 + x_3) + \tan\left(\frac{\alpha}{2}\right)(y_2 - y_3)}{2}, \frac{(y_2 + y_3) - \tan\left(\frac{\alpha}{2}\right)(x_2 - x_3)}{2} \right)$$

$$T = \left(\frac{(x_4 + x_1) + \tan\left(\frac{\alpha}{2}\right)(y_4 - y_1)}{2}, \frac{(y_4 + y_1) - \tan\left(\frac{\alpha}{2}\right)(x_4 - x_1)}{2} \right)$$

Now,

$$\text{slope of } PR = \left(\frac{(y_1 + y_2 - y_3 - y_4) - \cot\left(\frac{\alpha}{2}\right)(x_1 - x_2 - x_3 + x_4)}{(x_1 + x_2 - x_3 - x_4) + \cot\left(\frac{\alpha}{2}\right)(y_1 - y_2 - y_3 + y_4)} \right) = \frac{K_v}{L_v}(let)$$

$$\text{slope of } QT = \left(\frac{(y_2 + y_3 - y_4 - y_1) - \tan\left(\frac{\alpha}{2}\right)(x_2 - x_3 - x_4 + x_1)}{(x_2 + x_3 - x_4 - x_1) + \tan\left(\frac{\alpha}{2}\right)(y_2 - y_3 - y_4 + y_1)} \right) = \frac{M_v}{N_v}(let)$$

$$= -\left(\frac{(x_1 + x_2 - x_3 - x_4) + \cot\left(\frac{\alpha}{2}\right)(y_1 - y_2 - y_3 + y_4)}{(y_1 + y_2 - y_3 - y_4) - \cot\left(\frac{\alpha}{2}\right)(x_1 - x_2 - x_3 + x_4)} \right) = -\frac{L_v}{K_v}$$

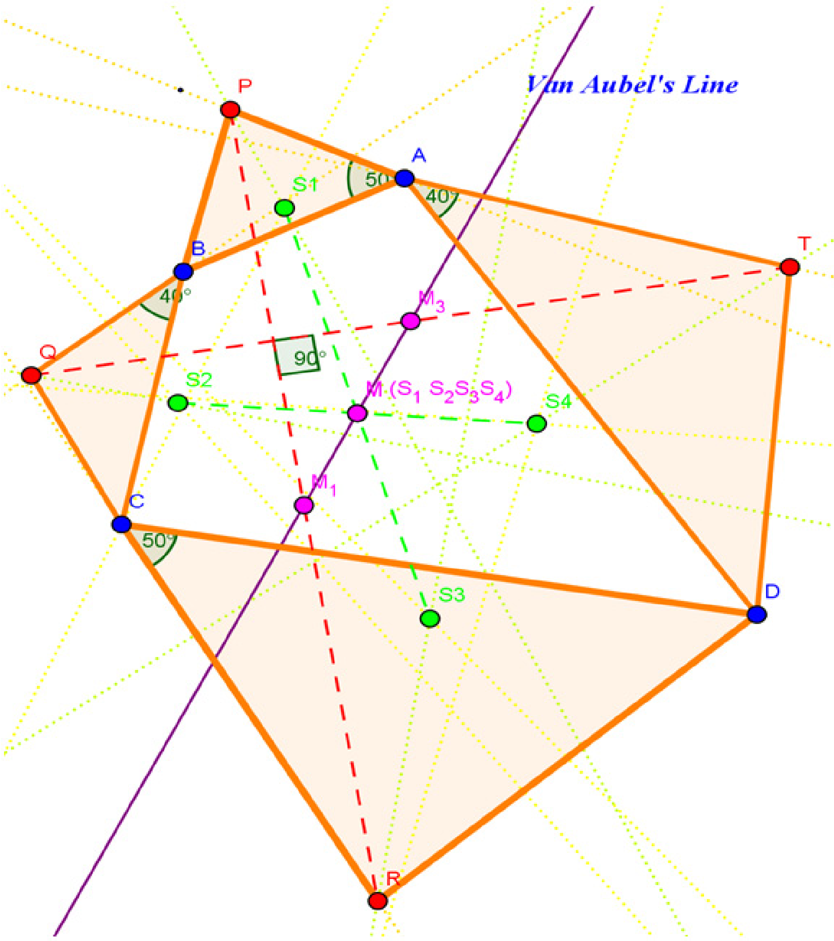


Fig. 8.

Here, it is clear that (slope of PR)(slope of QT) = −1

$$\Rightarrow M_v K_v + L_v N_v = 0 \tag{4}$$

That is $PR \perp QT$, Hence (a) is proved, Now for (b), Consider $K_v = \left(y_1 + y_2 - y_3 - y_4\right) - \cot\ (\alpha/2)\ (x_1 - x_2 - x_3 + x_4)$, $L_v = (x_1 + x_2 - x_3 - x_4) + \cot\ (\alpha/2)\ \left(y_1 - y_2 - y_3 + y_4\right)$, $M_v = \left(y_2 + y_3 - y_4 - y_1\right) - \tan\ (\alpha/2)\ (x_2 - x_3 - x_4 + x_1)$, $N_v = (x_2 + x_3 - x_4 - x_1) - \tan\ (\alpha/2)\ \left(y_2 - y_3 - y_4 + y_1\right)$

It is clear that $K_v = \cot\ (\alpha/2)\, N_v$ and $L_v = -\cot\ (\alpha/2)\, M_v$,
From (4), it is clear that

$$\frac{M_v}{N_v} = -\frac{L_v}{K_v} \Rightarrow \sqrt{\left(\frac{{L_v}^2 + {K_v}^2}{{M_v}^2 + {N_v}^2}\right)} = \frac{K_v}{N_v} = \frac{-L_v}{M_v} = \cot(\alpha/2)$$

Now

$$\left|\frac{PR}{QT}\right| = \left|\sqrt{\left(\frac{{L_v}^2 + {K_v}^2}{{M_v}^2 + {N_v}^2}\right)}\right| = \left|\frac{K_v}{N_v}\right| = \left|\frac{-L_v}{M_v}\right| = |\cot(\alpha/2)|$$

That is the ratio of two segments PR and QT doesn't depend on the quadrilateral. Hence (b) is proved.
Now for (c), we proceed as follows:

Since the base angles of isosceles triangles $\Delta ABP, \Delta CDR$ are $90^o - \alpha/2$, So, using 2.3, we have

$$S_1 = \left(\frac{(x_1 + x_2) + \cot\alpha\left(y_1 - y_2\right)}{2}, \frac{\left(y_1 + y_2\right) - \cot\alpha\,(x_1 - x_2)}{2}\right)$$

and

$$S_3 = \left(\frac{(x_3 + x_4) + \cot\alpha\left(y_3 - y_4\right)}{2}, \frac{\left(y_3 + y_4\right) - \cot\alpha\,(x_3 - x_4)}{2}\right)$$

In the similar manner, since the base angles of isosceles triangles $\Delta BCQ, \Delta DAT$ are $\alpha/2$. So, using 2.3, we have

$$S_2 = \left(\frac{(x_2+x_3) - \cot\alpha\,(y_2-y_3)}{2}, \frac{(y_2+y_3) + \cot\alpha\,(x_2-x_3)}{2}\right)$$

and

$$S_4 = \left(\frac{(x_4+x_1) - \cot\alpha\,(y_4-y_1)}{2}, \frac{(y_4+y_1) + \cot\alpha\,(x_4-x_1)}{2}\right)$$

Now, It is clear that, The mid point of S_1S_3 =The midpoint of S_2S_4

$$= \left(\frac{(x_1+x_2+x_3+x_4) + \cot\alpha\,(y_1-y_2+y_3-y_4)}{4}, \frac{(y_1+y_2+y_3+y_4) - \cot\alpha\,(x_1-x_2+x_3-x_4)}{4}\right)$$

Hence, the quadrilateral $S_1S_2S_3S_4$ is parallelogram, which completes the proof of (c).

Now for (d), Since the quadrilateral $S_1S_2S_3S_4$ is parallelogram, Van Aubel's point $\left(M_{S_1S_2S_3S_4}\right)$ of quadrilateral $S_1S_2S_3S_4$ is the midpoint of the diagonals.

Hence Van Aubel's point $\left(M_{S_1S_2S_3S_4}\right)$ of quadrilateral $S_1S_2S_3S_4$

$$M_{S_1S_2S_3S_4} = \left(\frac{(x_1+x_2+x_3+x_4) + \cot\alpha\,(y_1-y_2+y_3-y_4)}{4}, \frac{(y_1+y_2+y_3+y_4) - \cot\alpha\,(x_1-x_2+x_3-x_4)}{4}\right)$$

and Mid point of PR

$$M_{PR} = \left(\frac{(x_1+x_2+x_3+x_4) + \cot\left(\frac{\alpha}{2}\right)(y_1-y_2+y_3-y_4)}{4}, \frac{(y_1+y_2+y_3+y_4) - \cot\left(\frac{\alpha}{2}\right)(x_1-x_2+x_3-x_4)}{4}\right)$$

Mid point of QT

$$M_{QT} = \left(\frac{(x_1+x_2+x_3+x_4) - \tan\left(\frac{\theta}{2}\right)(y_1-y_2+y_3-y_4)}{4}, \frac{(y_1+y_2+y_3+y_4) + \tan\left(\frac{\theta}{2}\right)(x_1-x_2+x_3-x_4)}{4}\right)$$

Consider $\lambda = x_1+x_2+x_3+x_4$, $\beta = y_1+y_2+y_3+y_4$, $\gamma = x_1-x_2+x_3-x_4$ and $\delta = y_1-y_2+y_3-y_4$, then

$$M_{S_1S_2S_3S_4} = \left(\frac{\lambda + \cot\alpha\,\delta}{4}, \frac{\beta - \cot\alpha\,\gamma}{4}\right)$$

$$M_{PR} = \left(\frac{\lambda + \cot\left(\frac{\alpha}{2}\right)\delta}{4}, \frac{\beta - \cot\left(\frac{\alpha}{2}\right)\gamma}{4}\right)$$

$$M_{QT} = \left(\frac{\lambda - \tan\left(\frac{\alpha}{2}\right)\delta}{4}, \frac{\beta + \tan\left(\frac{\alpha}{2}\right)\gamma}{4}\right)$$

The midpoint of M M_{PR} and M_{QT}

$$= \left(\frac{2\lambda + \left(\cot\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\alpha}{2}\right)\right)\delta}{8}, \frac{2\beta - \left(\cot\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\alpha}{2}\right)\right)\gamma}{8}\right)$$

$$\left(\because \left(\cot\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\alpha}{2}\right)\right) = 2\cot\alpha\right)$$

$$= \left(\frac{\lambda + \cot\alpha\,\delta}{4}, \frac{\beta - \cot\alpha\,\gamma}{4}\right) = M_{S_1S_2S_3S_4}$$

Hence, the midpoint of $PR\,(M_{PR})$ and $QT\,\left(M_{QT}\right)$ is the Van Aubel's point $S_1S_2S_3S_4$.

Now consider a line (2), that is $4\gamma x + 4\delta y = \lambda\gamma + \beta\delta$. Clearly, the three points lies on this line (2). Hence, the three points $M_{S_1S_2S_3S_4}, M_{PR}, M_{QT}$ are collinear. The line through these points is $4\gamma x + 4\delta y = \lambda\gamma + \beta\delta$.

From Theorem 3.7, Clearly, the line (2) is van aubel's line (L). □

Theorem 3.12.

Let ABCD is a quadrilateral, suppose the triangles $\Delta ABP'$, $\Delta CDR'$, are isosceles with angle Îś at their top vertices, and $\Delta BCQ'$, $\Delta DAT'$ are isosceles with angle $\pi - \alpha$ at their top vertices (all of them have same orientation) constructed on the sides of the quadrilateral which lie inwards and if S_1', S_2', S_3', S_4' the circumcenters of the triangles $\Delta ABP'$, $\Delta BCQ'$, $\Delta CDR'$ and $\Delta DAT'$, Then *(a) P'R' is perpendicular to Q'T'*

(b) The ratio of these two segements, P'R' and Q'T', doesn't depend from the quadrilateral.

(c) Quadrilateral $S'_1,\ S'_2, S'_3, S'_4$ is parallelogram

(d) The three points, the Van Aubel's point $\left(M'_{S'_1,\ S'_2,S'_3,S'_4}\right)$ of quadrilateral $S'_1,\ S'_2, S'_3, S'_4$ and mid points of P'R'($M'_{P'R'}$) and Q'T'($M'_{Q'T'}$) are collinear and lie on the line Van Aubel's Line given by

$$4(x_1-x_2+x_3-x_4)x+4(y_1-y_2+y_3-y_4)y=(x_1+x_3)^2-(x_2+x_4)^2+\left(y_1+y_3\right)^2-\left(y_2+y_4\right)^2$$

And in particular the midpoint of P'R'($M'_{P'R'}$) and Q'T'($M'_{Q'T'}$) is the Van Aubel's point of $S_1,\ S_2, S_3, S_4$

Proof. Given at the top vertices P', R' makes an angle Îś, so the two isosceles triangles $\Delta ABP', \Delta CDR'$ having the base angle as $90^o - \alpha/2$. Using 2.1, we have

$$P' = \left(\frac{(x_1+x_2)-\cot\left(\frac{\alpha}{2}\right)\left(y_1-y_2\right)}{2}, \frac{\left(y_1+y_2\right)+\cot\left(\frac{\alpha}{2}\right)(x_1-x_2)}{2}\right)$$

$$R' = \left(\frac{(x_3+x_4)-\cot\left(\frac{\alpha}{2}\right)\left(y_3-y_4\right)}{2}, \frac{\left(y_3+y_4\right)+\cot\left(\frac{\alpha}{2}\right)(x_3-x_4)}{2}\right)$$

And given at the top vertices Q', T' makes an angle $\pi-\alpha$, so the two isosceles triangles $\Delta BCQ', \Delta DAT'$ having the base angle as $\alpha/2$. Hence, using 2.2, we have

$$Q' = \left(\frac{(x_2+x_3)-\tan\left(\frac{\alpha}{2}\right)\left(y_2-y_3\right)}{2}, \frac{\left(y_2+y_3\right)+\tan\left(\frac{\alpha}{2}\right)(x_2-x_3)}{2}\right)$$

$$T' = \left(\frac{(x_4+x_1)-\tan\left(\frac{\alpha}{2}\right)\left(y_4-y_1\right)}{2}, \frac{\left(y_4+y_1\right)+\tan\left(\frac{\alpha}{2}\right)(x_4-x_1)}{2}\right)$$

Now

$$\text{Slope of } P'R' = \left(\frac{\left(y_1+y_2-y_3-y_4\right)+\cot\left(\frac{\alpha}{2}\right)(x_1-x_2-x_3+x_4)}{(x_1+x_2-x_3-x_4)-\cot\left(\frac{\alpha}{2}\right)(y_1-y_2-y_3+y_4)}\right) = \frac{K_v'}{L_v'}(let)$$

$$\text{Slope of } Q'T' = \left(\frac{\left(y_2+y_3-y_4-y_1\right)+\tan\left(\frac{\alpha}{2}\right)(x_2-x_3-x_4+x_1)}{(x_2+x_3-x_4-x_1)-\tan\left(\frac{\alpha}{2}\right)(y_2-y_3-y_4+y_1)}\right) = \frac{M_v'}{N_v'}(let)$$

$$= -\left(\frac{(x_1+x_2-x_3-x_4)-\cot\left(\frac{\alpha}{2}\right)(y_1-y_2-y_3+y_4)}{\left(y_1+y_2-y_3-y_4\right)+\cot\left(\frac{\alpha}{2}\right)(x_1-x_2-x_3+x_4)}\right) = -\frac{L_v'}{K_v'}$$

It is clear that (slope of $P'R'$)(slope of $Q'T'$) $= -1$

$$\Rightarrow M_v'K_v' + L_v'N_v' = 0 \tag{5}$$

That is $P'R' \perp Q'T'$ Hence (a) is proved.
Now for (b), Consider

$$K_v' = \left(y_1+y_2-y_3-y_4\right)+\cot(\alpha/2)(x_1-x_2-x_3+x_4), L_v' = (x_1+x_2-x_3-x_4)-\cot(\alpha/2)\left(y_1-y_2-y_3+y_4\right)$$
$$M_v' = \left(y_2+y_3-y_4-y_1\right)+\tan(\alpha/2)(x_2-x_3-x_4+x_1), N_v' = (x_2+x_3-x_4-x_1)+\tan(\alpha/2)\left(y_2-y_3-y_4+y_1\right)$$

It is clear that

$$K_v' = -\cot(\alpha/2)\, N_v'$$

$K_v' = -\cot(\alpha/2)\, N_v'$ and $L_v' = \cot(\alpha/2)\, M_v'$, From (5), it is clear that

$$\frac{M_v'}{N_v'} = -\frac{L_v'}{K_v'} \Rightarrow \sqrt{\left(\frac{L_v'^2+K_v'^2}{M_v'^2+N_v'^2}\right)} = \frac{K_v'}{N_v'} = \frac{-L_v'}{M_v'} = -\cot(\alpha/2)$$

Now

$$\left|\frac{P'R'}{Q'T'}\right| = \left|\sqrt{\left(\frac{L'^2+K'^2}{M'^2+N'^2}\right)}\right| = \left|\frac{K'}{N'}\right| = \left|\frac{-L'}{M'}\right| = |-\cot(\alpha/2)|$$

That is the ratio of two segments $P'R'$ and $Q'T'$ doesn't depend from the quadrilateral. Hence (b) is proved.
Now for (c), we proceed as follows:

Since the base angles of isosceles triangles $\Delta ABP', \Delta CDR'$ are $90^o - \alpha/2$ So, using 2.3, we have

$$S'_1 = \left(\frac{(x_1+x_2)-\cot\alpha\,(y_1-y_2)}{2}, \frac{(y_1+y_2)+\cot\alpha\,(x_1-x_2)}{2}\right)$$

and

$$S'_3 = \left(\frac{(x_3+x_4)-\cot\alpha\,(y_3-y_4)}{2}, \frac{(y_3+y_4)+\cot\alpha\,(x_3-x_4)}{2}\right)$$

In the similar manner, since the base angles of isosceles triangles $\Delta BCQ', \Delta DAT'$ are $\alpha/2$. So using 2.3, we have

$$S'_2 = \left(\frac{(x_2+x_3)+\cot\alpha\,(y_2-y_3)}{2}, \frac{(y_2+y_3)-\cot\alpha\,(x_2-x_3)}{2}\right)$$

and

$$S'_4 = \left(\frac{(x_4+x_1)+\cot\alpha\,(y_4-y_1)}{2}, \frac{(y_4+y_1)-\cot\alpha\,(x_4-x_1)}{2}\right)$$

Now, It is clear that, The mid point of $S'_1S'_3$ =The midpoint of $S'_2S'_4$ =

$$= \left(\frac{(x_1+x_2+x_3+x_4)-\cot\alpha\,(y_1-y_2+y_3-y_4)}{4}, \frac{(y_1+y_2+y_3+y_4)+\cot\alpha\,(x_1-x_2+x_3-x_4)}{4}\right)$$

Hence, the quadrilateral $S'_1S'_2S'_3S'_4$ is parallelogram, which completes the proof of (c).

Now for (d), Since the quadrilateral $S'_1S'_2S'_3S'_4$ is parallelogram, Van Aubel's point $\left(M_{S'_1S'_2S'_3S'_4}\right)$ of quadrilateral $S'_1S'_2S'_3S'_4$ is the midpoint of the diagonals. Hence, Van Aubel's point $\left(M_{S'_1S'_2S'_3S'_4}\right)$ of quadrilateral $S'_1S'_2S'_3S'_4$

$$M'_{S'_1S'_2S'_3S'_4} = \left(\frac{(x_1+x_2+x_3+x_4)-\cot\alpha\,(y_1-y_2+y_3-y_4)}{4}, \frac{(y_1+y_2+y_3+y_4)+\cot\alpha\,(x_1-x_2+x_3-x_4)}{4}\right)$$

And midpoint of $P'R'$ =

$$M'_{P'R'} = \left(\frac{(x_1+x_2+x_3+x_4)-\cot\left(\frac{\alpha}{2}\right)(y_1-y_2+y_3-y_4)}{4}, \frac{(y_1+y_2+y_3+y_4)+\cot\left(\frac{\alpha}{2}\right)(x_1-x_2+x_3-x_4)}{4}\right)$$

Mid point of $Q'T'$ =

$$M'_{Q'T'} = \left(\frac{(x_1+x_2+x_3+x_4)+\tan\left(\frac{\alpha}{2}\right)(y_1-y_2+y_3-y_4)}{4}, \frac{(y_1+y_2+y_3+y_4)-\tan\left(\frac{\alpha}{2}\right)(x_1-x_2+x_3-x_4)}{4}\right)$$

Consider $\lambda = x_1+x_2+x_3+x_4, \beta = y_1+y_2+y_3+y_4, \gamma = x_1-x_2+x_3-x_4, \delta = y_1-y_2+y_3-y_4$, then

$$M'_{S'_1S'_2S'_3S'_4} = \left(\frac{\lambda-\cot\alpha\,\delta}{4}, \frac{\beta+\cot\alpha\,\gamma}{4}\right)$$

$$M'_{P'R'} = \left(\frac{\lambda-\cot\left(\frac{\alpha}{2}\right)\delta}{4}, \frac{\beta+\cot\left(\frac{\alpha}{2}\right)\gamma}{4}\right)$$

$$M'_{Q'T'} = \left(\frac{\lambda+\tan\left(\frac{\alpha}{2}\right)\delta}{4}, \frac{\beta-\tan\left(\frac{\alpha}{2}\right)\gamma}{4}\right)$$

The midpoint of $M'_{P'R'}$ and $M'_{Q'T'}$

$$= \left(\frac{2\lambda-\left(\cot\left(\frac{\alpha}{2}\right)-\tan\left(\frac{\alpha}{2}\right)\right)\delta}{8}, \frac{2\beta+\left(\cot\left(\frac{\alpha}{2}\right)-\tan\left(\frac{\alpha}{2}\right)\right)\gamma}{8}\right)$$

$$= \left(\frac{\lambda-\cot\alpha\,\delta}{4}, \frac{\beta+\cot\alpha\,\gamma}{4}\right) = M'_{S'_1S'_2S'_3S'_4} \quad \left(\because \left(\cot\left(\frac{\alpha}{2}\right)-\tan\left(\frac{\alpha}{2}\right)\right) = 2\cot\alpha\right)$$

Hence, the midpoint of $P'R'\,(M'_{P'R'})$ and $Q'T'\left(M'_{Q'T'}\right)$ is the Van Aubel's point of $S'_1S'_2S'_3S'_4$
Now consider a line (2), that is $4\gamma x + 4\delta y = \lambda\gamma + \beta\delta$. Clearly, the three points lies on this line (2). Hence, the three points $M_{S'_1S'_2S'_3S'_4}, M'_{P'R'}, M'_{Q'T'}$ are collinear. The line through these points is $4\gamma x + 4\delta y = \lambda\gamma + \beta\delta$.

From Theorem 3.7, Clearly, the line (2) is van aubel's line (L). □

Remark 3.4. 1. The Van Aubel's point $\left(M_{S_1S_2S_3S_4}\right)$ of quadrilateral $S_1S_2S_3S_4$ the Van Aubel's point $\left(M'_{S'_1S'_2S'_3S'_4}\right)$ of quadrilateral $S'_1S'_2S'_3S'_4$ and the four points $M_{PR}, M_{QT}, M'_{P'R'}, M'_{Q'T'}$ are collinear lie on the Van Aubel's Line.

2. Using 3.2, 3.3 it is clear that Van Aubel's line contains 15 points, They are Van Aubel's point (M) of the quadrilateral $ABCD$, the points $M_1, M_2, M_3, M_4, M'_1, M'_2, M'_3, M'_4$, the Van Abel's point of quadrilateral $S_1S_2S_3S_4$, the Van Abel's point of quadrilateral $S'_1S'_2S'_3S'_4$, $M_{PR}, M_{QT}, M'_{P'R'}, M'_{Q'T'}$.

Theorem 3.13.
The following statements are true.

(a) The Van Aubel's point (M) of quadrilateral ABCD is the midpoint of (Van Aubel 's points of the quadrilaterals $S_1S_2S_3S_4$ and $S'_1S'_2S'_3S'_4$), $(M_{PR}, M'_{P'R'})$ and $\left(M_{QT}, M'_{Q'T'}\right)$ (see Fig. 9)

(b) $M_{PR}M'_{Q'T'} = M_{S_1S_2S_3S_4}M'_{S'_1S'_2S'_3S'_4} = M_{QT}M'_{P'R'}$

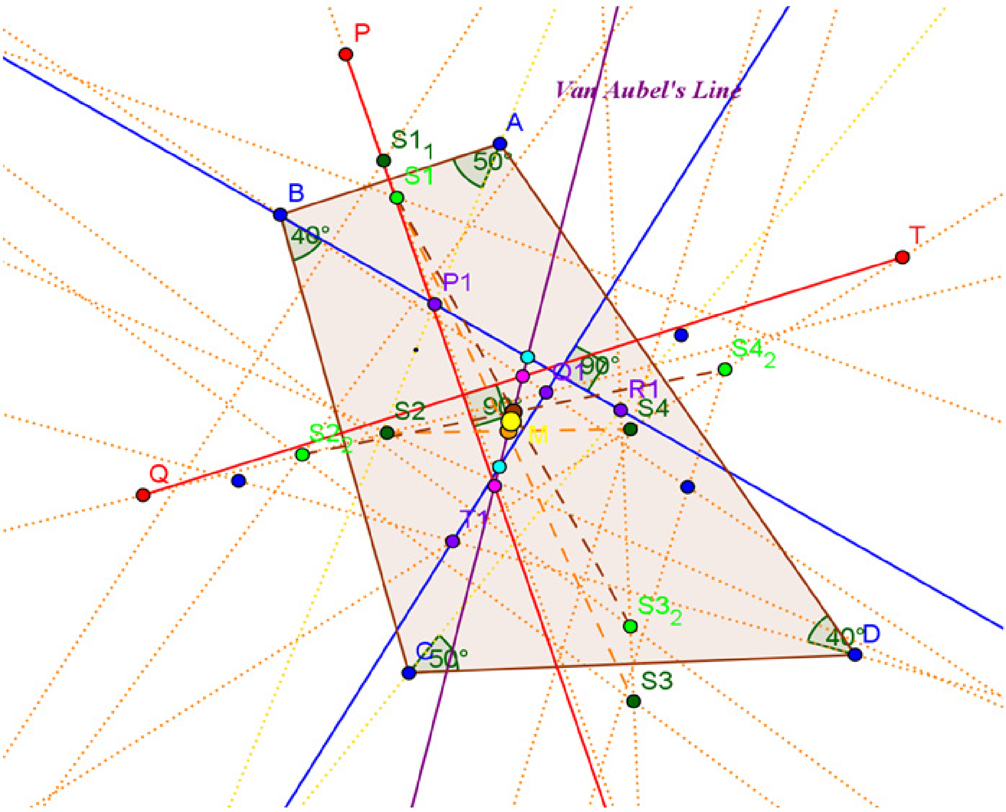


Fig. 9.

Proof. We have

$$M_{S_1S_2S_3S_4} = \left(\frac{\lambda + \cot\alpha\,\delta}{4}, \frac{\beta - \cot\alpha\,\gamma}{4}\right)$$

$$M_{PR} = \left(\frac{\lambda + \cot\left(\frac{\alpha}{2}\right)\delta}{4}, \frac{\beta - \cot\left(\frac{\alpha}{2}\right)\gamma}{4}\right)$$

$$M_{QT} = \left(\frac{\lambda - \tan\left(\frac{\alpha}{2}\right)\delta}{4}, \frac{\beta + \tan\left(\frac{\alpha}{2}\right)\gamma}{4}\right)$$

$$M'_{S'_1S'_2S'_3S'_4} = \left(\frac{\lambda - \cot\alpha\,\delta}{4}, \frac{\beta + \cot\alpha\,\gamma}{4}\right)$$

$$M'_{P'R'} = \left(\frac{\lambda - \cot\left(\frac{\alpha}{2}\right)\delta}{4}, \frac{\beta + \cot\left(\frac{\alpha}{2}\right)\gamma}{4}\right)$$

$$M'_{Q'T'} = \left(\frac{\lambda + \tan\left(\frac{\alpha}{2}\right)\delta}{4}, \frac{\beta - \tan\left(\frac{\alpha}{2}\right)\gamma}{4}\right)$$

Now it is clear that the Van Aubel's point (M) of quadrilateral $ABCD$ is the midpoint of the (Van Aubel's points of the quadrilaterals $S_1S_2S_3S_4$ and $S'_1S'_2S'_3S'_4$), $(M_{PR}, M'_{P'R'})$, $\left(M_{QT}, M'_{Q'T'}\right)$.
Hence (a) is proved.
Now for (b), Consider

$$M_{PR}M'_{Q'T'} = \left|\left(\frac{\cot\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\alpha}{2}\right)}{4}\right)\right|\left(\sqrt{\delta^2+\gamma^2}\right) = \left|\frac{\cot\alpha}{2}\right|\left(\sqrt{\delta^2+\gamma^2}\right)$$

$$M_{S_1S_2S_3S_4}M'_{S'_1S'_2S'_3S'_4} = \left|\frac{\cot\alpha}{2}\right|\left(\sqrt{\delta^2+\gamma^2}\right)$$

$$M_{QT}M'_{P'R'} = \left|\left(\frac{\cot\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\alpha}{2}\right)}{4}\right)\right|\left(\sqrt{\delta^2+\gamma^2}\right) = \left|\frac{\cot\alpha}{2}\right|\left(\sqrt{\delta^2+\gamma^2}\right)$$

Hence

$$M_{PR}M'_{Q'T'} = M_{S_1S_2S_3S_4}M'_{S'_1S'_2S'_3S'_4} = M_{QT}M'_{P'R'}$$

□

3.2. Dao's Generalization [1]

Theorem 3.14.
Let ABCD be a quadrilateral, let four points A_1, B_1, C_1, D_1 on the plane either interior or exterior to the quadrilateral such that $\angle A_1AB = \angle DAD_1 = \alpha$, $\angle B_1BC = \angle ABA_1 = \beta$, $\angle BCB_1 = \angle C_1CD = \gamma$, $\angle CDC_1 = \angle D_1DA = \delta$ and $\alpha + \gamma = \beta + \delta = \frac{\pi}{2}$ in the same

(a) $A_1B_1C_1D_1$ is an orthodiagonal quadrilateral.

(b) The ratio of these two segments, A_1C_1 and B_1D_1 doesn't depend from the quadrilateral.

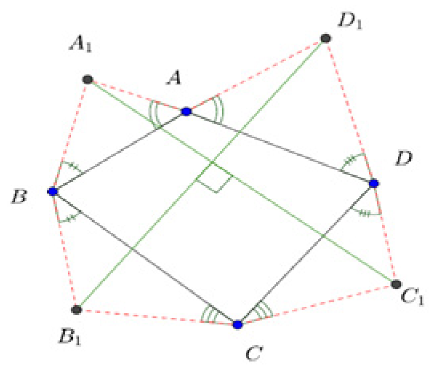


Fig. 10.

Proof. Without loss of generality let us consider the coordinates of vertices of the quadrilateral $ABCD$ as $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$ and $D = (x_4, y_4)$ and also given $\angle A_1AB = \angle DAD_1 = \alpha, \angle B_1BC = \angle ABA_1 = \beta, \angle BCB_1 = \angle C_1CD = \gamma = 90^0 - \alpha, \angle CDC_1 = \angle D_1DA = \delta = 90^0 - \beta$. So using 2.1, we have the coordinates of A_1, B_1, C_1, D_1 as follows

$$A_1 = \left(\frac{(x_1\cot\beta + x_2\cot\alpha) \pm (y_1 - y_2)}{\cot\alpha + \cot\beta}, \frac{(y_1\cot\beta + y_2\cot\alpha) \mp (x_1 - x_2)}{\cot\alpha + \cot\beta}\right)$$

$$B_1 = \left(\frac{(x_2\cot\gamma + x_3\cot\beta) \pm (y_2 - y_3)}{\cot\beta + \cot\gamma}, \frac{(y_2\cot\gamma + y_3\cot\beta) \mp (x_2 - x_3)}{\cot\beta + \cot\gamma}\right)$$

$$= \left(\frac{(x_2 + x_3\cot\alpha\cot\beta) \pm \cot\alpha\,(y_2 - y_3)}{1 + \cot\alpha\cot\beta}, \frac{(y_2 + y_3\cot\alpha\cot\beta) \mp \cot\alpha\,(x_2 - x_3)}{1 + \cot\alpha\cot\beta}\right) \quad (\text{since } \gamma = 90^0 - \alpha)$$

$$C_1 = \left(\frac{\left(x_3\cot\delta + x_4\cot\gamma\right) \pm \left(y_3 - y_4\right)}{\cot\gamma + \cot\delta}\ ,\ \frac{\left(y_3\cot\delta + y_4\cot\gamma\right) \mp (x_3 - x_4)}{\cot\gamma + \cot\delta}\right)$$

$$= \left(\frac{\left(x_3\cot\alpha + x_4\cot\beta\right) \pm \cot\alpha\cot\beta\left(y_3 - y_4\right)}{\cot\alpha + \cot\beta}, \frac{\left(y_3\cot\alpha + y_4\cot\beta\right) \mp \cot\alpha\cot\beta\,(x_3 - x_4)}{\cot\alpha + \cot\beta}\right) \left(\text{since } \gamma = 90^0 - \alpha,\ \delta = 90^0 - \beta\right)$$

$$D_1 = \left(\frac{(x_4\cot\alpha + x_1\cot\delta) \pm \left(y_4 - y_1\right)}{\cot\delta + \cot\alpha}\ ,\ \frac{\left(y_4\cot\alpha + y_1\cot\delta\right) \mp (x_4 - x_1)}{\cot\delta + \cot\alpha}\right)$$

$$= \left(\frac{\left(x_4\cot\alpha\cot\beta + x_1\right) \pm \cot\beta\left(y_4 - y_1\right)}{1 + \cot\alpha\cot\beta}, \frac{\left(y_4\cot\alpha\cot\beta + y_1\right) \mp \cot\beta\,(x_4 - x_1)}{1 + \cot\alpha\cot\beta}\right) \quad \left(\text{since } \delta = 90^0 - \beta\right)$$

Now

$$\text{Slope of } A_1C_1 = \frac{\cot\alpha(y_3 - y_2) + \cot\beta(y_4 - y_1) \mp \left[\cot\alpha\cot\beta(x_3 - x_4) - (x_1 - x_2)\right]}{\cot\alpha(x_3 - x_2) + \cot\beta(x_4 - x_1) \pm \left[\cot\alpha\cot\beta(y_3 - y_4) - (y_1 - y_2)\right]} = \frac{K_d}{L_d}$$

$$\text{Slope of } B_1D_1 = \frac{(y_1 - y_2) + (y_4 - y_3)\cot\alpha\cot\beta \mp \left[\cot\beta(x_4 - x_1) - \cot\alpha(x_2 - x_3)\right]}{(x_1 - x_2) + (x_4 - x_3)\cot\alpha\cot\beta \pm \left[\cot\beta(y_4 - y_1) - \cot\alpha(y_2 - y_3)\right]} = \frac{M_d}{N_d}$$

$$= -\left(\frac{\cot\alpha(x_3 - x_2) + \cot\beta(x_4 - x_1) \pm \left[\cot\alpha\cot\beta(y_3 - y_4) - (y_1 - y_2)\right]}{\cot\alpha(y_3 - y_2) + \cot\beta(y_4 - y_1) \mp \left[\cot\alpha\cot\beta(x_3 - x_4) - (x_1 - x_2)\right]}\right) = -\frac{L_d}{K_d}$$

So, it is clear that (Slope of A_1C_1)(Slope of B_1D_1)= -1,

$$\Rightarrow M_dK_d + L_dN_d = 0 \tag{6}$$

Hence $A_1C_1 \perp B_1D_1$, that is quadrilateral $A_1C_1B_1D_1$ is orthodiagonal quadrilateral. So, (a) is proved.
Now for (b), Consider

$$K_d = \cot\alpha(y_3 - y_2) + \cot\beta(y_4 - y_1) \mp \left[\cot\alpha\cot\beta(x_3 - x_4) - (x_1 - x_2)\right]$$

$$L_d = \cot\alpha(x_3 - x_2) + \cot\beta(x_4 - x_1) \pm \left[\cot\alpha\cot\beta(y_3 - y_4) - (y_1 - y_2)\right]$$

$$M_d = (y_1 - y_2) + (y_4 - y_3)\cot\alpha\cot\beta \mp \left[\cot\beta(x_4 - x_1) - \cot\alpha(x_2 - x_3)\right]$$

$$N_d = (x_1 - x_2) + (x_4 - x_3)\cot\alpha\cot\beta \pm \left[\cot\beta(y_4 - y_1) - \cot\alpha(y_2 - y_3)\right]$$

It is clear that $K_d = N_d$ and $L_d = -M_d$.
From (6), it is clear that

$$\frac{M_d}{N_d} = -\frac{L_d}{K_d} \Rightarrow \sqrt{\left(\frac{{L_d}^2 + {K_d}^2}{{M_d}^2 + {N_d}^2}\right)} = \frac{K_d}{N_d} = \frac{-L_d}{M_d} = 1$$

Now

$$\left|\frac{A_1C_1}{B_1D_1}\right| = \left|\frac{\cot\alpha\cot\beta + 1}{\cot\alpha + \cot\beta}\right|\sqrt{\left(\frac{{L_d}^2 + {K_d}^2}{{M_d}^2 + {N_d}^2}\right)} = \left|\left(\frac{\cot\alpha\cot\beta + 1}{\cot\alpha + \cot\beta}\right)\frac{K_d}{N_d}\right| = \left|-\left(\frac{\cot\alpha\cot\beta + 1}{\cot\alpha + \cot\beta}\right)\frac{L_d}{M_d}\right| = \left|\frac{\cot\alpha\cot\beta + 1}{\cot\alpha + \cot\beta}\right|$$

That is the ratio of two segments PR and QT doesn't depend from the quadrilateral. Hence (b) is proved. □

3.3. Generalization of Kiepert Hyperbola

Suppose the diagonals of quadrilaterals $ABCD$ are equal, suppose the triangles $\Delta ABP, \Delta CDR$ are isosceles with angle ψ at their top vertices, and the triangles $\Delta BCQ, \Delta DAT$ are isosceles with angle $\pi - \psi$ at their top vertices (all of them have the same orientation). Then intersection of PR and QT moves along an equilateral hyperbola passing through the midpoints of diagonals and asymptotes midlines of the quadrilateral. For further generalizations refer [2].

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