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Using dynamic geometry to expand mathematics teachers' understanding of proof[1]

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This paper gives a broad descriptive account of some activities that the author has designed using *Sketchpad* to develop teachers' understanding of other functions of proof than just the traditional function of 'verification'. These other functions of proof illustrated here are those of explanation, discovery and systematization (in the context of defining and classifying some quadrilaterals). A solid theoretical rationale is provided for dealing with these other functions in teaching by analysing actual mathematical practice where verification is not always the most important function. The activities are designed according to the so-called 'reconstructive' approach, and are structured more or less in accordance with the Van Hiele theory of learning geometry.

1. Introduction

Traditionally, teachers have presented proof in the geometry classroom only as a means of obtaining certainty; i.e. to try to create doubts in the minds of their students about the validity of their empirical observations, and thereby attempting to motivate a need for deductive proof. This approach stems possibly from a narrow formalist view that the main function of proof is the verification of the correctness of mathematical statements. It has also dominated most mathematics teacher education courses. In fact, in 1987 the author found that about 60% of prospective secondary school mathematics teachers at South African universities saw the function of proof only in terms of verification/justification/conviction [1], and were not able to distinguish any other functions of proof.

However, proof has many other important functions within mathematics, which in some situations are of far greater importance than that of mere verification. Some of these are [2–4]:

- explanation (providing insight into **why** it is true)
- discovery (the discovery or invention of **new** results)
- intellectual challenge (the **self-realization/fulfilment** derived from constructing a proof)

[1] This is an extended and refined version of a paper originally presented at the Working Group on The Use of Technology in Mathematics Education, ICME-9, July 2000, Tsukuba, Japan.

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- systematization (the **organization** of various results into a deductive system of axioms, concepts and theorems)

Traditionally teachers try to raise students' doubts and then attempt to introduce proof as a means of 'making sure'. This is often problematic within a dynamic geometry context because students typically obtain an extremely high level of confidence through continuous manipulation. Furthermore, it is also problematic from a mathematical point of view, since it is mostly not doubt that motivates the finding of a proof, but conviction of the truth of the result. Several personal examples of these during the discovery of new results (and rediscovery of old results) are given in [3, 5, 6].

In addition, as pointed out in [7], most of the traditional strategies used in geometry education to create doubt are highly questionable. Frequently, these examples are from number theory (sometimes cleverly disguised as geometry) and therefore do not correspond to the continuous variation of variables in a dynamic geometry context. Otherwise, they are based on optical illusions or false proofs (based on incorrectly drawn diagrams). In both cases, a need for proof as a means of verification is hardly created. Instead, it actually encourages empirical verification, since accurate construction and measurement is usually needed to resolve the illusion or false proof.

Several other authors [8–10] have also previously argued, from both epistemological and psychological viewpoints, that mathematics teachers should in their own teaching also attempt to develop an understanding and appreciation of these other functions to make proof a more meaningful activity for their students. Teachers ought to induct their own students early into the art of problem posing, and allowing sufficient opportunity for exploration, conjecturing, refuting, reformulating and explaining. However, if teachers themselves have never been exposed to such approaches in their own learning of mathematics, it is hardly likely that they would attempt to implement it in their own classrooms (compare Polya [11, p. 173; 12, pp. 115–119]). It therefore seems fundamentally important in mathematics teacher education to devise ways of expanding prospective teachers' views of proof, and to allow sufficient opportunity for exploration, conjecturing and explaining.

Traditionally most teachers and textbook authors have simply provided students with ready-made content (e.g. definitions, theorems, proofs, classifications, etc.) that they merely have to assimilate and regurgitate in tests and exams. Traditional geometry education of this kind can be compared to a cooking and bakery class where the teacher only shows students cakes (or even worse, only pictures of cakes) without showing them what goes into the cake and how it is made. In addition, they're not even allowed to try their own hand at baking!

The purpose of this paper will be to present and discuss some examples of *Sketchpad* activities which the author has developed in mathematics teacher education courses to specifically focus on expanding students' understanding of several different functions of proof. Although this paper will focus only on the explanation, discovery and systematization functions of proof, other functions such as verification and intellectual challenge are also incorporated in this approach [4, 13, 14]. Some of these activities have also successfully been used with junior high school students [15] and mathematics teacher education courses at another training institution in KwaZulu-Natal [16].

The approach is closely linked to the Van Hiele theory [17] of which a brief outline will be given. In addition, it is based on a 'reconstructive' teaching approach, which means that the content is not directly presented to students in a finished form, but is re-constructed by the learners and the teacher in a typical mathematical manner [18]. A theoretical distinction will also be made between two different kinds of defining, namely, descriptive and constructive defining of concepts.

2. The role and function of proof

In actual mathematical practice, conviction is far more frequently a prerequisite for the finding of a proof than doubt (compare [19], pp. 83, 84). Doug Hofstadter [20, p. 10], for example, also emphasizes as follows how conviction within a dynamic geometry context can precede, and motivate, a proof:

> By the way, note that I just referred to my screen-based observation as a ''fact'' and a ''theorem''. Now any redblooded mathematician would scream murder at me for referring to a ''fact'' or ''theorem'' that I had not proved. But that is not my attitude at all, and never has been. To me, this result was so clearly true that I didn't have the slightest doubt about it. I didn't need proof. If this sounds arrogant, let me explain. The beauty of Geometer's Sketchpad is that it allows you to discover instantly whether a conjecture is right or wrong – if it's wrong, it will be immediately obvious when you play around with a construction dynamically on the screen. If it's right, things will stay ''in synch'' right on the button no matter how you play with the figure. The degree of certainty and confidence that this gives is downright amazing. It's not a proof, of course, but in some sense, I would argue, this kind of direct contact with the phenomenon is even more convincing than a proof, because you really see it all happening right before your eyes. None of this means that I did not want a proof. In the end, proofs are critical ingredients of mathematical knowledge, and I like them as much as anyone else does. I am just not one who believes that certainty can come **only** from proofs.

With the above it is not meant to disregard the importance of proof as a valuable means of verification, especially in the case of surprising, non-intuitive or doubtful results. Indeed the author believes that one still needs to attempt to address the verification function of proof within a dynamic geometry context as illustrated by activities [4, pp. 71–95]. However, care should be taken that the examples are authentic, and not contrived, and that the students themselves actually experience real doubt, expressing a genuine need for conviction [21, 22].

A major problem with only empirical verification (whether it is by hand or electronic device) not often highlighted, is that it gives no psychological satisfactory sense of illumination, i.e. an insight or understanding into how it is the consequence of other familiar results [23]. Even when one could actually check all cases for a particular result (e.g. when the number of cases are finite), this provides no understanding of why it is true. For example, *why* is it that if you take any 2-digit number and add it to the number with the same digits reversed, then the result is always divisible by 11? Although this can easily be checked in all cases, this is hardly satisfactory.

It should also be noted that to the working mathematician proof is not merely a means of *a posteriori* verification, but often also a means of discovery and invention [24]. For example, explaining (proving) why something is true quite often enables an immediate generalization that might not have been suggested at all by purely empirical exploration. Several example of these can be found [13, 5, 6]. However, new results can also be discovered *a priori* by simply logically analysing the properties of given mathematical objects. For example, without resorting to actual construction and measurement it is often possible to quickly logically deduce certain implied properties with the aid of a roughly drawn sketch.

Without proof it would also be impossible to organize the results of our mathematical research into a deductive system of axioms, definitions and theorems. In most such cases of *a posteriori* systematization [25], the purpose is not to check whether the results are really true, but to organize these logically unrelated results – which are already known to be true – into a coherent unified whole. For example, the primary function of a proof for the intermediate value theorem for continuous functions is that of systematization, as a simple picture combined with an informal argument is sufficient for the purposes of both verification and illumination (explanation).

3. The Van Hiele theory

The Van Hiele theory originated in the respective doctoral dissertations of Dina van Hiele-Geldof and her husband Pierre van Hiele at the University of Utrecht, Netherlands in 1957. While Pierre's dissertation mainly tried to explain why students experienced problems in geometry education (in this respect it was explanatory and descriptive), Dina's dissertation was about a teaching experiment (and in that sense was more prescriptive regarding the ordering of geometry content and learning activities of students). The most obvious characteristic of the theory is the distinction of five discrete thought levels in respect to the development of students' understanding of geometry.

According to the Van Hiele theory, the main reason for the failure of the traditional geometry curriculum at the high school is that the curriculum is presented at a higher level than those of the students; in other words they cannot understand the teacher nor can the teacher understand why they cannot understand! Although the Van Hiele theory distinguishes between five different levels of thought, we shall here only focus on the first four levels as they are the most pertinent ones for secondary school geometry. The general characteristics of each level can be described as follows:

Level 1: Recognition

Students visually recognize figures by their global appearance. They recognize triangles, squars, parallelograms, and so forth by their shape, but they do not explicitly identify the properties of these figures.

Level 2: Analysis

Students start analysing the properties of figures and learn the appropriate technical terminology for describing them, but they they do not interrelate figures or properties of figures.

Level 3: Ordering

Students logically order the properties of figures by short chains of deductions and understand the interelationships between figures (eg. class inclusions).

Level 4: Deduction

Students start developing longer sequences of statements and begin to understand the significance of deduction, the role of axioms, theorems and proof.

According to the Van Hiele theory, deductive reasoning first occurs on Level 3 when the network of logical relationships between properties is established. In other words, when a proof for the equality of the diagonals of a rectangle is developed, the meaning of such a proof lies in making the logical relationships between the properties explicit. A student at Level 1 or 2, who does not yet possess this network of logical implications, can only experience such a proof as an attempt at the verification of the result. However, since such students do not doubt the validity of their empirical observations, they tend to experience it as meaningless, i.e. 'proving the obvious'. It should further be noted that the transition from Van Hiele Level 1 to Level 2 poses specific problems to second language learners, since it also involves the acquisition of the technical terminology by which the properties of figures need to be described and explored. This is particularly important in a country like South Africa where large numbers of learners at school and university are taught in a language other than their mother tongue, the latter in many cases not having generic equivalents to the technical terminology being used.

According to the Van Hiele theory for learning to be meaningful, students should become acquainted with, and explore geometry content in phases which correspond to the Van Hiele Levels. A serious shortcoming of the Van Hiele theory, however, is that there is no explicit distinction between different possible functions (meanings) of proof. For example, the development of deductive thinking (proof) appears first within the context of systematization at Van Hiele Level 3 (Ordering). Empirical research [26, 27] seems to indicate, however, that the functions (meanings) of logical deduction (proof) in terms of explanation, discovery and verification can be meaningful to students outside a systematization context, in other words, at Van Hiele Levels lower than Van Hiele Level 3 (provided the arguments are of an intuitive or visual nature; e.g. the use of symmetry or dissection, etc.). From experience, it also seems that a prolonged delay at Van Hiele Levels 1 and 2 before an introduction to proof, actually makes the later introduction of proof as a meaningful activity even more difficult.

4. The Reconstructive Approach

The traditional practice of presenting mathematical topics as completed axiomatic-deductive systems has been strongly criticized by well-known advocates of the 'genetic' approach such as Wittmann [28], Polya [12] and Freudenthal [29]. Essentially, the genetic approach argues that the learner should either retrace (at least in part) the path followed by the original discoverers or inventors, or retrace a path by which mathematical content could have been discovered or invented. Human [30] calls it the 'reconstructive' approach and contrasts it with the so-called 'direct axiomatic-deductive' approach. Whereas in the latter, content is directly introduced to learners (as finished products of mathematical activity), learners in the former, have to newly reconstruct the content in a typical mathematical fashion.

A reconstructive approach typically highlights the meaning (actuality) of the content, and allows students to actively participate in the construction and the development of the content. In recent times, the learning theory of constructivism provides a valuable psychological perspective strongly supporting such a teaching approach. A reconstructive approach does not necessarily imply learning by discovery for it may just be a reconstructive explanation by the teacher or the textbook. It also does not mean that one strictly follows a historical approach, but simply that the history of mathematics serves as a useful guide.

5. Defining

Whereas axiomatization refers to the systematization of several results (theorems) into a mathematical system, (formal) defining refers to the systematization of the properties of a particular concept. Instead of actively engaging teachers in the former global activity, most of the initial activities I use focus on systematisation at the local level, i.e. defining concepts such as quadrilaterals, which provides a familiar, but still challenging context.

The direct teaching of geometry definitions with no emphasis on the process of defining has often been criticized by mathematicians and mathematics educators alike [29, pp. 416–418). Ohtani [31, pp. 81] has argued that the traditional practice of simply telling definitions to students is a method of moral persuasion with several social functions, amongst which are: to justify the teacher's control over the students; to attain a degree of uniformity; to avoid having to deal with students' ideas; and to circumvent problematic interactions with students. Vinner [32] and others make a useful distinction between the concept image and concept definition of a concept. Just knowing the definition of a concept does not at all guarantee understanding of the concept. For example, although a student may have been taught, and be able to recite, the standard definition of a parallelogram as a quadrilateral with opposite sides parallel, the student may still not consider rectangles, squares and rhombi as parallelograms, since the students' concept image of a parallelogram is one in which not all angles or sides are allowed to be equal.

Linchevsky *et al.* [33] further reported that many student teachers do not even understand that definitions in geometry have to be economical (contain no superfluous information) and that they are arbitrary (in the sense that several alternative definitions may exist). It is plausible to conjecture that this is due to their past school experiences where definitions were probably supplied directly to them. It would appear that in order to increase students' understanding of geometric definitions, and of the concepts to which they relate, it is essential to engage them at some stage in the process of defining of geometric concepts. Since defining concepts is not easy, it would also appear to be unreasonable to expect students to immediately come up with formal definitions on their own, unless they have been guided in a didactic fashion through some examples of the process of defining which they can later use as models for their own attempts.

It is useful to distinguish between two different types of defining of concepts, namely, descriptive (*a posteriori)* and constructive (*a priori*) defining [34].

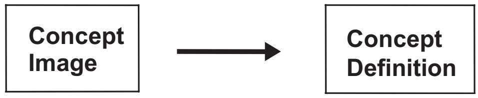

Figure 1.

Figure 2.

5.1. *Descriptive defining*

With the descriptive (*a posteriori*) defining of a concept is meant here that the concept and its properties have already been known for some time and is defined only afterwards. In other words, the concept image of the concept is already well developed before a concept definition is formulated for it (figure 1). *A posteriori* defining is usually accomplished by selecting an appropriate subset of the total set of properties of the concept from which all the other properties can be deduced. This subset then serves as the definition and the other remaining properties are then logically derived from it as theorems.

5.2. *Constructive defining*

Constructive *(a priori)* defining takes places when a given definition of a concept is changed through the exclusion, generalization, specialization, replacement or addition of properties to the definition, so that a new concept is constructed in the process (figure 3). In other words, a new concept is defined 'into being', the further properties of which can then be experimentally or logically explored. In this case, the concept definition of the new concept precedes the later exploration and further development of its concept image (figure 2). Whereas the main purpose or function of *a posteriori* defining is that of the systematisation of existing knowledge, the main function of *a priori* defining is the production of new knowledge.

Speaking from a historical perspective, as well as personal experience, most concepts are usually defined in a descriptive manner although constructive defining has become a more prominent process in the last century. Furthermore, although these two ways of defining can be distinguished from one another, they are not necessarily disjoint processes. Sometimes they go hand in hand and are intertwined in the generation of a particular definition.[2]

6. Sample activities

The activities below have been developed and refined over the past 25 years, starting with the University of Stellenbosch Experiment with Mathematics

[2] Rasmussen and Zandieh [39] give an example of where these two defining processes appear to be intertwined. Here students adapted/generalized their definitions for straight lines and triangles of the plane to the sphere after an initial, informal experiential exploration of these concepts on the sphere.

Education (USEME) conducted in 1977/1978 [35]. The activities have been gradually refined through implementation and evaluation in high school classrooms ranging from Grades 9–12, as well as working with teachers in both a pre-service and in-service capacity. Specifically, the sample of activities below are currently being used in a final year (4th year) Special Methods Course in Mathematics Education for prospective secondary school mathematics teachers, as well as with post-graduate students (mostly practising teachers) at the University of Durban-Westville, South Africa. The students in this course would normally have completed at least four semester courses in mathematics, or at least two semester courses in mathematics together with six semester courses in mathematics education.

The University of Durban-Westville is historically an apartheid university which was originally established in the 1960s for Indian students only, but currently the student body consists mainly of black and Indian students, in about equal proportions. The majority of black students tend to be strongly disadvantaged in geometry due to the fact that many mathematics teachers in black township schools are uncomfortable with geometry,[3] and except for a few 'book' theorems, hardly do any geometry in their classes.

Whereas the traditional approach to Euclidean geometry in South Africa focuses overridingly on developing the ability of making deductive proofs (especially for riders), these activities are aimed mainly at:

- providing the prospective mathematics teachers with exemplars of how geometric content could be organized in learning activities corresponding to the Van Hiele levels;
- developing understanding of varied meanings or roles of proof at the different levels (e.g. explanation, discovery, systematization, etc.);
- actively engaging the prospective mathematics teachers in the process of defining in order that they may realize (1) that different, alternative definitions for the same concept are possible, (2) that definitions may be uneconomical or economical, (3) that some economical definitions lead to shorter, easier proofs of properties;
- developing the prospective mathematics teachers' ability to construct formal, economical definitions for geometrical concepts.

Less familiar concepts such as an isosceles trapezoid and kite are specifically used, since these are quadrilaterals for which the students are not likely to have

[3] Many reasons for this can be given. Due to the shortage of mathematics teachers generally, many black teachers currently teaching mathematics are under-qualified or unqualified to teach the subject. Such teachers tend to feel that high school algebra and calculus are much easier topics to teach as they believe it can be taught algorithmically, whereas the solution of typical riders in our high school geometry often requires far more creative thinking, and is not only more difficult for themselves to accomplish, but also to teach. However, even well-qualified teachers (in all population groups) often have difficulty with geometry. The reason is that in most teacher education institutions in South Africa, such as universities and colleges, there is a heavy focus on calculus and algebra in the mathematics courses, with hardly any geometry being done. So many return to teach only with high school geometry as their highest qualification in geometry, and so perpetuate the cycle.

encountered a formal definition nor studied at school. The defining of the isosceles trapezoid (see Activity 3 below) is dealt with first (usually with substantial assistance from the teacher), and this then acts as an exemplar for the students to tackle the defining of a kite more or less unassisted. Students are then subsequently required to develop similar activities for the more well-known quadrilaterals such as squares, parallelograms, rectangles, etc. Note that for obvious reasons the activities below have been greatly abbreviated and that the actual worksheets include a lot more technical detail and structure [4]. In the latter, use is also made of ready-made (or partially ready-made) sketches, which students can simply manipulate in order to utilize classroom time effectively. Although *Sketchpad* is used, the activities can obviously be done with any other dynamic geometry software, as well as in paper and pencil contexts (although not as effectively).

ACTIVITY 1: *Exploration of properties of an isosceles trapezoid*

Construct

(1) Construct a line and any line segment not on the line.
(2) Reflect the line segment in the line.
(3) Connect corresponding points to obtain a quadrilateral as shown in figure 3.

Investigate

(1) Can you make any conjectures regarding the following properties of the type of figure in figure 3? Check your conjectures by dragging.
    (a) sides
    (b) angles
    (c) diagonals
    (d) inscribed or circumscribed circle.
(2) Can the above figures sometimes be a parallelogram, rectangle, rhombus or square? Investigate by dragging.
(3) Can you logically explain your conjectures in terms of symmetry?

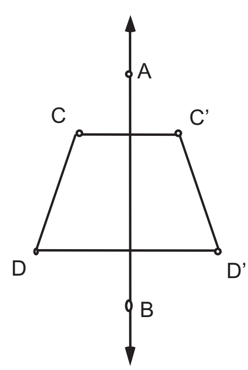

Figure 3.

Discussion

In this activity students use *Sketchpad* to first construct an isosceles trapezoid by using reflection, and then explore its properties by measurement and/or construction (e.g. angles, sides, diagonals, circum circle). It firstly involves Van Hiele Level 1 (visualization) as students are encouraged to drag it into different shapes and orientations. It also involves Van Hiele Level 2 (analysis and formulation of properties) as students measure properties and make conjectures. Students find it relatively easy to come up with conjectures like the following, and display very high levels of confidence having explored them by dragging:

(1) One pair of equal opposite sides and another pair of parallel sides.
(2) Two pairs of equal adjacent angles.
(3) Equal diagonals.
(4) Perpendicular bisectors are concurrent; therefore it is cyclic. (The students normally need some assistance to discover this conjecture).

Students are also asked to drag their figure into the crossed case shown in figure 4 and to investigate whether these conjectures also hold there. Note, however, that although all of the above conjectures apply to the crossed isosceles trapezoid, Sketchpad shows all four angles equal. (If we have already at this stage investigated crossed quadrilaterals, their interior angle sum, and developed a working definition for its 'interior' angles, its two pairs of equal adjacent angles can be identified as shown in figure 5.)

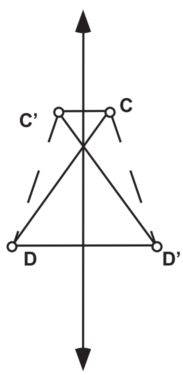

Figure 4.

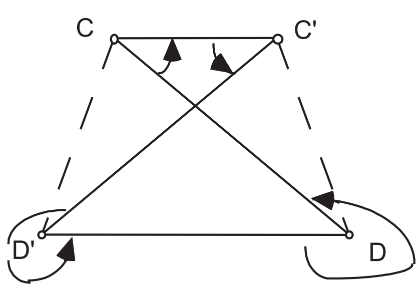

Figure 5.

In Question 2 above, students investigate whether their constructed figure can be dragged into special quadrilaterals. Although hierarchical class inclusion[4] of quadrilaterals is supposed to occur only at Van Hiele Level 3, it seems that most students have little difficulty accepting a rectangle and a square as special cases. In addition, (general) parallelograms and rhombi are recognized as not being special cases, since the figure cannot be dragged into those shapes. In fact, it generally seems that the dynamic nature of geometric figures constructed and explored in *Sketchpad* make it possible for students to accept and understand hierarchical class inclusions already at Van Hiele Level 1 (Visualization), but further research into this particular area is needed.

In Question 3, the properties of an isosceles trapezoid are then explained by the students in terms of reflective symmetry (in some cases with assistance), for example: CD maps onto C′D′ and therefore one pair of opposite sides are equal, angles C and D respectively map onto angles C′ and D′ and therefore two pairs of adjacent angles are equal, CC′ and DD′ are both perpendicular to the line of symmetry and are therefore parallel to each other, CD′ upon reflection around AB maps it onto C′D and therefore diagonals are equal.

Interestingly, students mostly do not use symmetry to explain Conjecture 4 above (cyclic property), but usually show that the opposite angles are supplementary (which is a typical way in high school geometry of proving that a quadrilateral is cyclic). They therefore find the following argument when presented to them something of an eye-opener (due to the realization that there may actually be other ways of proving a quadrilateral cyclic than the ones they were taught at school).

Consider the isosceles trapezoid in figure 6. From symmetry, it follows that both perpendicular bisectors of AD and BC coincide with the axis of symmetry. The perpendicular bisector of AB intersects the axis of symmetry at O. But since AB maps onto DC around the axis of symmetry, its perpendicular bisector must also map onto the perpendicular bisector of DC (and vice versa), implying that these two perpendicular bisectors must intersect each other at O. Since the perpendicular bisectors are concurrent, there exists a point equidistant from all four vertices and a circle can be drawn with O as center and AO as radius.

ACTIVITY 2: *Constructing midpoints of sides of isosceles trapezoid*

Construct and connect the midpoints of the sides of an isosceles trapezoid as shown in figure 7.

(1) Investigate the type of quadrilateral formed by the midpoints of its sides.
(2) Can you logically explain your conjecture?

[4] Although most Van Hiele literature places hierarchical class inclusion at Level 3, Van Hiele [17, p. 93] argues as follows that it could occur at Level 2 (analysis of properties):

> The development of a network of relations results in a rhombus becoming a symbol for a large set of properties. The relationship of the rhombus to other figures is now determined by this collection of properties. Students who have progressed to this level, will answer the question of what a rhombus is by saying: 'A rhombus is a quadrilateral with four equal sides, with opposite angles equal and with perpendicular bisecting diagonals which also bisect the angles.' On the grounds of this, a square now becomes a rhombus. (freely translated from the Dutch)

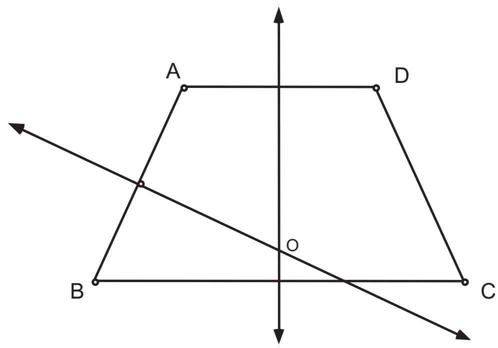

Figure 6.

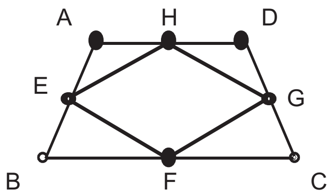

Figure 7.

(3) From Question 2, can you find/construct another more general type of quadrilateral that will have the same midpoint property?

Discussion

The main purpose of this activity is to introduce students to the discovery function of proof discussed earlier. In this activity, students construct the midpoints of the sides of a dynamic isosceles trapezoid, and explore the kind of figure formed, leading to the eventual conjecture that it is always a rhombus (or a square as a special case). Students are again asked to also consider whether this is also true for the crossed case. The students are also asked to explore, and explain the conditions under which the figure becomes a square. The following is an example of the kind of explanation students come up with after been given the hint to look at the diagonals and using the triangle midpoint theorem:

EF and HG are equal to ½ AC and EH and FG are equal to ½ BD. But diagonals AC = BD implies that all the sides of EFGH are equal; ie. it is a rhombus. Further, if AC⊥BD, then the two pairs of opposite sides of EFGH are perpendicular on each other; i.e. all the angles are equal, which means that the rhombus becomes a square.

In Question 3, students are asked to carefully consider which property they used to explain why it is a rhombus, and whether they could use this property (of equal diagonals) to further generalize the result (by constructing a general quadrilateral with equal diagonals on *Sketchpad*). To make the generalization, students have to first realize that equal diagonals is a necessary, but not sufficient condition for a quadrilateral to be an isosceles trapezoid. (This activity therefore lies in a transition phase from Van Hiele Level 2 to Van Hiele Level 3.) Students tend to find this question rather difficult and usually need to be given some guidance to make a suitable construction. A quadrilateral with equal diagonals can

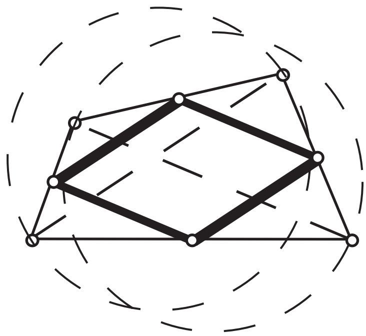

Figure 8.

be constructed by first constructing a line segment, and then two circles with equal diameters with the endpoints of the line segments respectively lying on the two circles (figure 8). The above construction can be used to drag the quadrilateral into convex, concave and crossed cases. Afterwards the point is made that this general result involving a quadrilateral with equal diagonals is unlikely to have been discovered purely by chance, and that the explanation for the isosceles trapezium assisted us in this generalization by identifying the essential characteristic which ensured that the inscribed figure was always a rhombus.

ACTIVITY 3: *Describing (defining) an isosceles trapezoid*

The isosceles trapezoid has the following properties:

(a) (At least) one line of symmetry through a pair of opposite sides.
(b) Equal diagonals.
(c) (At least) one pair of opposite sides parallel.
(d) (At least) one pair of opposite sides equal.
(e) Two (distinct) pairs of adjacent angles equal.
(f) Cyclic.

(1) How would you over the phone (or via telegram) explain what these quadrilaterals are to someone not yet acquainted with them? (Try to keep your description as short as possible, but ensure that the person has enough information to make a correct drawing of the quadrilateral).
(2) Test your description by making a construction on Sketchpad according to the conditions contained in it, and test it by dragging to see if it always remains an isosceles trapezoid.
(3) Try formulating two other alternative descriptions, and test by means of an appropriate construction. Which of the three do you like best? Why?
(4) Can you prove, using only logical deduction, that all the other properties of an isosceles trapezoid listed above, but not included in your three descriptions, can be derived from it? Which of the three do you still like best? Why?

Discussion

The main purpose of this activity is to develop understanding of the systematization function of proof. In other words, that proof is a valuable tool in the organization of known results into a deductive system of definitions and theorems. The focus here is not to explain or verify whether these properties are true or not, but to investigate their underlying logical relationships, as well as different possible systematisations. Note that this activity is at Van Hiele Level 3, since it involves descriptive defining (based on an analysis of the logical relationships between the properties).

A further purpose is to introduce students to a mathematical definition as an economical, but accurate description of an object. The purpose here is not for students to produce a single correct, economical definition, but to engage them in the activity of trying out various possibilities. Responses to the various questions vary considerably, and the students usually need substantial guidance the first time round. They tend to initially make many mistakes, i.e. including too many properties or too little, but only in making such mistakes do they gradually learn what defining is all about.

Note that making constructions as in Question 2 to test their descriptions (definitions) helps them to see the inter-relationships between properties, namely that some properties imply others; a fundamental characteristic of Van Hiele Level 3. In other words, these constructions are extremely important in that they develop explicit understanding of 'if-then' relationships, i.e. if we construct a cyclic quadrilateral with equal diagonals, then it will have (at least) one axis of symmetry, (at least) one pair of opposite sides parallel, (at least) one pair of opposite sides equal, etc. Smith [36] also reported marked improvement in students' understanding of 'if–then' statements by letting them make constructions to evaluate geometric statements as follows:

> Pupils saw that when they did certain things in making a figure, certain other things resulted. They learned to feel the difference in category between the relationships they put into a figure – the things over which they had control – and the relationships which resulted without any action on their part. Finally the difference in these two categories was associated with the difference between the given conditions and conclusion, between the if-part and the then-part of a sentence.

Even though the context of using a phone (or telegram) is deliberately used to encourage the students to keep their descriptions (definitions) as short as possible (to save cost), most of them tend to initially provide uneconomical descriptions (definitions); in other words, containing redundant information. The following is a typical example: 'An isosceles trapezoid is any quadrilateral with (at least) one axis of symmetry through a pair of opposite sides, and one pair of adjacent angles equal'. Asking them to construct a figure corresponding to this definition often makes them realize that 'one pair of equal adjacent angles' is redundant; they do not use it in their construction (reflection), and is a consequence of that construction. In general, it is pointed out to students that a correct, but uneconomical description (definition) that contains too much information can be improved by leaving out some of the properties. However, they must still ensure that the conditions are sufficient. For example, what happens if we define an isosceles trapezoid as any quadrilateral with one pair of adjacent angles equal? Clearly this definition is incorrect, since it is possible to construct a quadrilateral with one pair of adjacent angles equal which is not an isosceles trapezoid (figure 9). Having students make constructions like these to test their own or each others definitions

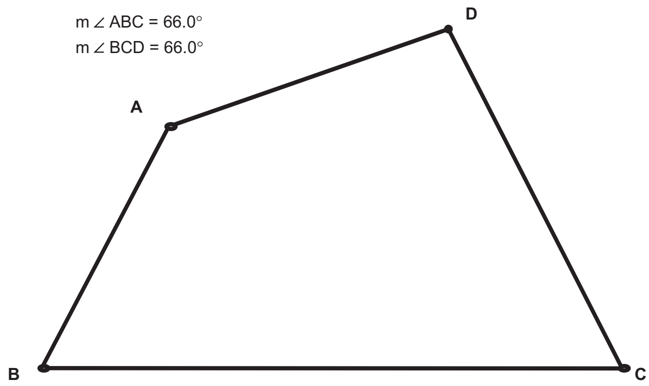

Figure 9.

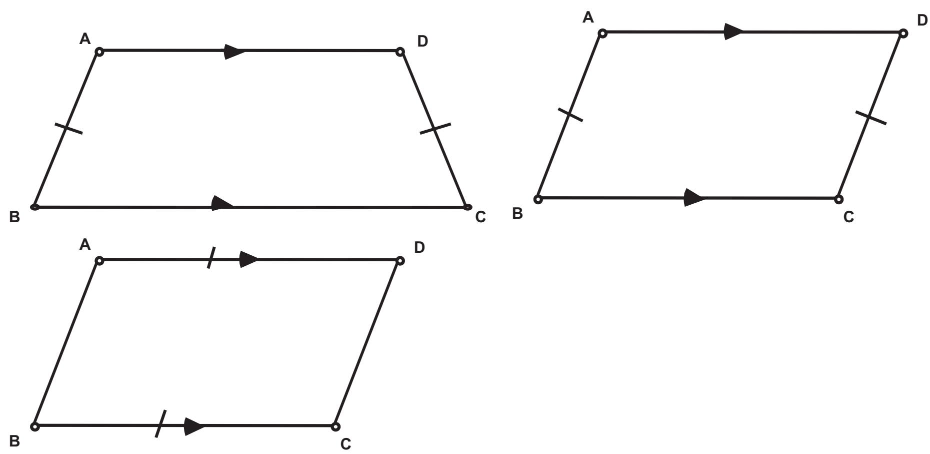

Figure 10.

therefore creates a natural context for a brief introductory discussion of necessary and sufficient conditions.

A popular initial choice of definition among students is the following (which incidentally appears in some textbooks): 'An isosceles trapezoid is any quadrilateral with (at least) one pair of parallel sides and (at least) one pair of opposite sides equal'.

However, this definition is unacceptable as it is ambiguous. For example, although one can obtain an isosceles trapezoid as shown in the first figure in figure 10, it also includes parallelograms as shown by the other two figures. It is usually necessary to explain that a (general) parallelogram cannot be considered an isosceles trapezium as it does not have all the properties of an isosceles trapezium (e.g. equal diagonals, two pairs adjacent angles equal, cyclic, axis of symmetry, etc.).

Although students usually attempt to improve this definition, there is no satisfactory way of correcting it. If they formulate it in such a way as to exclude the parallelograms, e.g. 'An isosceles trapezoid is any quadrilateral with one pair of opposite sides parallel, and another pair of opposite sides equal, but not parallel', they not only exclude the (general) parallelogram cases 2 and 3 above, but also the rectangles and squares. But as seen earlier in the first activity, students had already begun to consider the rectangles and squares as special cases of the isosceles trapezoids through dragging. It is then pointed out to them that although such partition definitions (which exclude special cases) are not mathematically incorrect [37], this tends to be very uneconomical. For example, the inclusive (hierarchical) approach is economical, since any theorems we may prove for isosceles trapezoids (e.g. equal diagonals) then automatically apply to rectangles and squares, and there is no need to repeat them for the rectangles and squares. In addition, partition definitions are generally longer than inclusive ones as they have to include more information (to ensure that special cases are excluded).

Apart from students' own descriptions, some possible definitions are also suggested to them to consider. These are usually a mixture of incomplete, uneconomical, or correct economical definitions, as well as definitions that require subtle modification, for example: 'An isosceles trapezoid is any quadrilateral with at least one pair of opposite sides parallel, and equal diagonals'.

Checking this definition by construction and dragging on *Sketchpad* shows that it works fine for the convex case (figure 11), but in the crossed case one could get a crossed quadrilateral as shown by the second figure in figure 12 which is not an isosceles trapezium. Strictly speaking we therefore have to specify that the equal diagonals have to be non-parallel. (Also note that this crossed quadrilateral is not a parallelogram since one pair of opposite sides (AB and CD) are not parallel, but intersect.)

It should also be noted that some sketches for correct definitions like 'An isosceles trapezoid is any cyclic quadrilateral with (at least) one pair of opposite parallel' do not allow for the dynamic transformation from a convex to a crossed case, and instead becomes a triangle. This is due to technical problems with *Sketchpad* (in the way in which it handles intersections of objects). This particular problem, however, does not occur with *Cabri*.

In Question 4, students are expected to systematize the properties of an isosceles trapezoid in different ways. For example, by starting from a given definition, they have to deduce the other properties from it as theorems. The following is an example.

Definition. An isosceles trapezoid is any cyclic quadrilateral with at least one pair of opposite sides parallel.

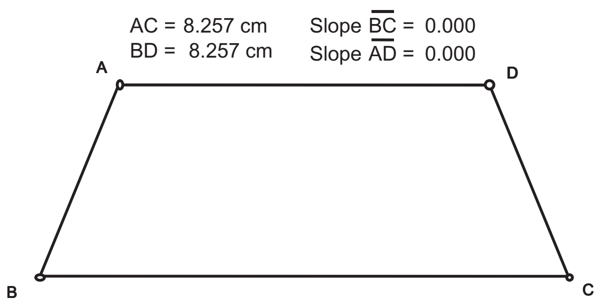

Figure 11.

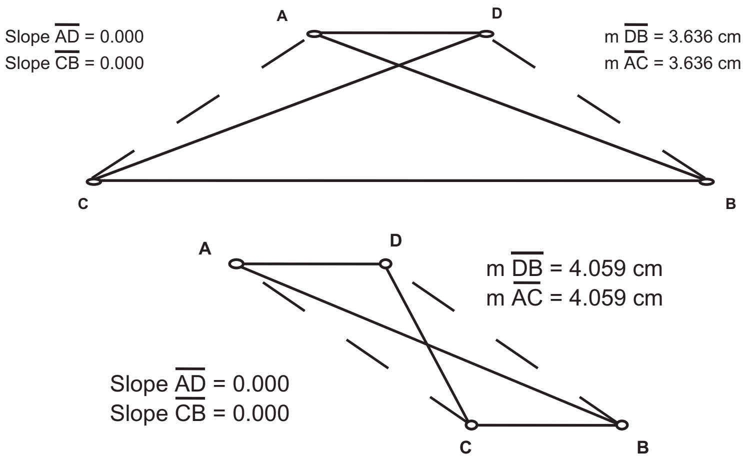

Figure 12.

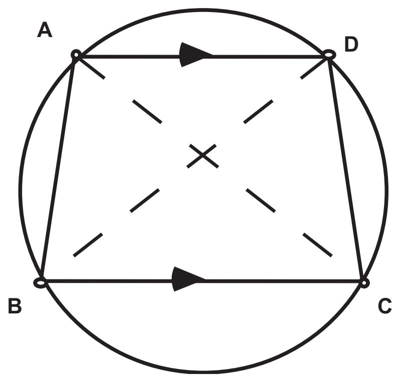

Figure 13.

(We shall here restrict ourselves to the convex CYCLIC case, but similar proofs can be given for the crossed cyclic case).

Theorem 1. An isosceles trapezoid has two (distinct) pairs of adjacent angles equal.

Proof. Consider figure 13 where it is given that ABCD is cyclic and AD//BC. Angles A and C are supplementary since ABCD is cyclic. Angles A and B are also supplementary, since they are co-interior with AB the tranversal to the parallels AD and BC. Therefore angle B = angle C. In the same way follows that angle A = angle D.

Theorem 2. An isosceles trapezoid has equal diagonals.

Proof. Consider the same figure above. From Theorem 1, we have angle ABC = angle DCB. But since equal angles are subtended by equal chords, it follows that the two chords subtending these two angles must be equal; i.e. AC = DB.

Continuing in the same way, students prove that an isosceles trapezoid has at least one pair of opposite sides equal, as well as an axis of symmetry. The students are then required to repeat the same systematization exercise for at least another two correct definitions from the following list that they would have formulated or identified by now on the basis of construction and measurement:

(1) An isosceles trapezoid is any quadrilateral with at least one pair of parallel sides and equal diagonals.
(2) An isosceles trapezoid is any quadrilateral with (at least one) axis of symmetry through a pair of opposite sides.
(3) An isosceles trapezoid is any cyclic quadrilateral with (at least) one pair of opposite sides equal.
(4) An isosceles trapezoid is any cyclic quadrilateral with (at least) one pair of adjacent angles equal.
(5) An isosceles trapezoid is any quadrilateral with two (distinct) pairs of adjacent angles equal.
(6) An isosceles trapezoid is any quadrilateral with equal diagonals and (at least) one pair of opposite sides equal.

In Questions 3 and 4, students are also asked to compare some different definitions and their corresponding systematizations, and to choose with reasons the one the prefer most. It is interesting to note that students mostly prefer the definition of an isosceles trapezoid in terms of its axis of symmetry through a pair of opposite sides, their reason usually being that it provides the simplest way of deducing the remaining properties from it, i.e. they virtually all follow directly from the symmetry property. In contrast, the other definitions require certain constructions or the use of longer or more complicated arguments. In other words, according to them a 'good' definition of a concept is one which allows one to easily deduce the other properties of a concept, i.e. it should be deductive-economical.

Some students have also occasionally said that they prefer the 'axis of symmetry' definition because it provides the easiest way of constructing it on *Sketchpad*. In other words, another aspect by which one could compare different definitions is whether or not a particular definition allows one to directly construct the object being defined. For example, defining an isosceles trapezoid as any cyclic quadrilateral with at least one pair of opposite sides equal, allows one to easily construct it. However, defining it as any quadrilateral with two distinct pairs of adjacent angles equal does not allow one to directly construct it using only the properties contained in the definition. The former definition could be called a constructable definition, whereas the latter could be called a non-constructable definition. It seems customary (although not always the case) that mathematicians also generally prefer constructable definitions, and that their preferred choice of 'an axis of symmetry' definition satisfies this criterion as well.

ACTIVITY 4: *Generalizing or Specializing*

(1) Generalize the concept isosceles trapezoid in different ways by leaving out some of its properties.
(2) Specialize the concept isosceles trapezoid in different ways by the addition of more properties.

Discussion

This activity probably corresponds best with Van Hiele Level 4 as students are required here to generalize or specialize the concept of an isosceles trapezoid, and work more formally with definitions. It is intended as a reconstructive demonstration of the mathematical process of constructive defining whereby new objects are defined by the modification or extension of the definitions of known objects. By encouraging students to generalize to other quadrilaterals by leaving out some of its properties, they usually manage to arrive at most of the possibilities shown in figure 14. Even though most of them do not necessarily have interesting properties, it provides a good examplar to reflect on and discuss the general process of constructive defining in mathematics.

Students are also asked how they think an isosceles trapezoid can be generalized to other polygons. One possibility is an isosceles hexagon, namely, a hexagon with at least one axis of symmetry through a pair of opposite sides (figure 15). Students are then asked to investigate its properties on *Sketchpad*, make conjectures, prove them, and whether they can generalize any further. (In general, for any $2n$-gon ($n > 1$) the two sums of alternate angles are equal, and there are INT($n/2$) pairs of equal 'diagonals'. These follow directly from symmetry.)

Students are also asked to consider specializing the concept of an isosceles trapezoid by adding additional properties. Depending on the properties being added one can of course obtain either a rectangle or a square. One interesting possibility, however, is that of a trilateral trapezoid, namely, an isosceles trapezoid with at least three equal sides as shown in figure 16. Students are asked to investigate on *Sketchpad* whether it has any interesting additional properties, and to prove any conjectures they may make. (The diagonals DB and AC respectively bisect angles ABC and DCB.)

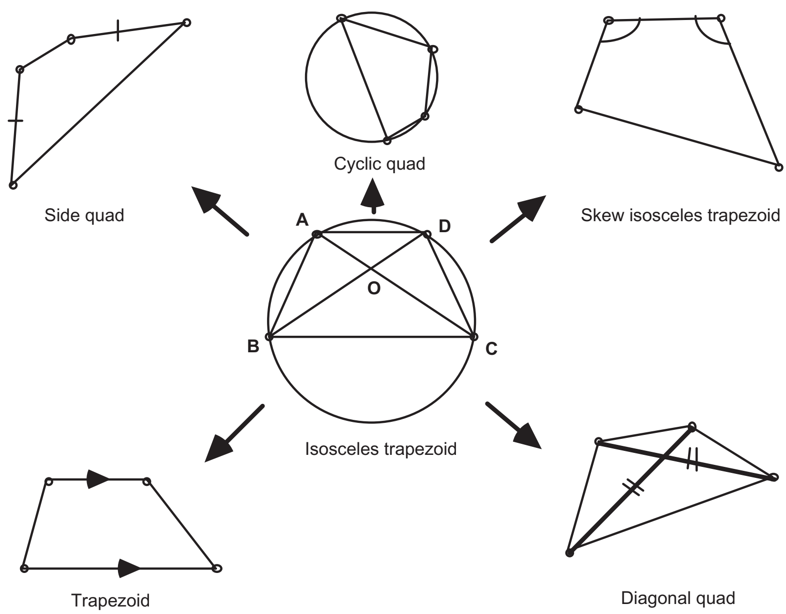

Figure 14.

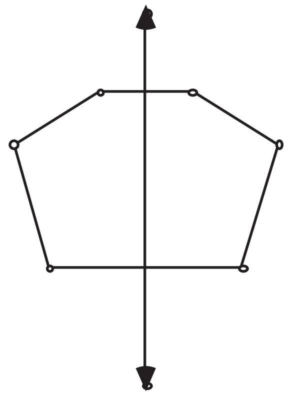

Figure 15.

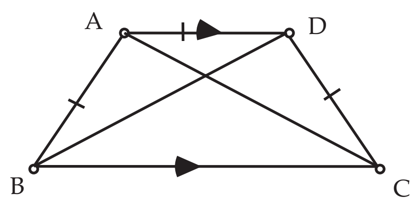

Figure 16.

7. Concluding comments

The reader is reminded that the concept of a kite is investigated alongside with that of an isosceles trapezoid. Although the prospective mathematics teachers (as well as the practising teachers!) usually require substantial assistance with the formal defining of an isosceles trapezoid (and its extensions and specialisations), far less assistance is needed when the same activities are repeated for a kite. This indicates that they do indeed develop some ability in the processes of descriptive and constructive defining.[5] In addition, several have commented that this reconstructive approach made it possible for them for the first time to see where mathematical definitions actually came from, how they are 'created', and that several different, correct alternatives could exist for the same concept. It therefore seems important for mathematics teachers (at least once) to be engaged in this kind of activity to develop a better understanding of the nature of definitions, as well as skill in defining objects on their own.

The structuring of the activities in levels corresponding to the Van Hiele levels appear to assist all the mathematics teachers (including those already practising), but in particular those who have had poor learning experiences in high school geometry themselves. This learning experience also provides them with the opportunity to reflect on the implications of the Van Hiele theory for planning and designing their own learning and teaching activities for the high school.

[5] In a reconstructive teaching approach in 1977/78 with Grade 10 high school pupils it was similarly found in a classic control/experimental group comparison that the experimental group had developed a higher ability to correctly, and economically, define an unknown object [18].

It also provides a valuable context to reflect on and discuss some rather neglected functions of proof such as explanation, discovery, and systematization, and how they could incorporate these in a high school geometry curriculum.

Finally, although several aspects of the reconstructive approach described here are currently being researched by myself and some of my students, it is ultimately based more on a philosophy rather than just empirical evidence. Freudenthal [38, pp. 43–44] writes similarly:

> I characterised my book 'Mathematics as an Educational Task' as a philosophy of mathematics education. I there explained the Socratic method and the method of reinvention: but I had no evidence to justify them other than pictures of man . . . I have not proved that what I aspire to is better, as little as one really knows whether teaching is more effective without beating – possibly it is not better. I am advocating another method because I believe in it, because I believe in the right of the learning child to be treated as a learning human being . . . the same freedom of trying and experimenting, of analysing before synthesising, the same right to integrate material, to make mistakes, to think provisionally and to acquire one's verbal expression by one's own efforts.

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