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Variations on a Japanese Temple theorem

1 Introduction

The following Japanese Temple geometry theorem appears in Fukagawa and Rigby [1]:

THEOREM 1. The products of the radii of the excircles on each pair of opposite sides of a circumscribed quadrilateral are equal.

See Figure 1. A *circumscribed* quadrilateral (also called a *tangential* quadrilateral) is one that possesses an inscribed circle. Now the diagram contains four lines (the sides of the quadrilateral) and one circle S (the inscribed circle) that touches all four of them; this we call the *base circle* of the theorem. The other four circles each touch just three of the lines. For the moment, assume no two of these lines are parallel, and also assume that no circle apart from S touches all four lines; the exceptional cases will be discussed in Section 4. So any three of the lines are sides of a triangle. Now there are four circles that touch the sides of a triangle, namely its incircle and three excircles, which we shall refer to as the *touch-circles* of the triangle, or of the three lines. Any three lines in Figure 1 are touched by the base circle and one other circle, so there are two "missing" touch-circles of the same three lines. If we put in all the missing circles, we shall have the base circle and twelve other circles, and the natural question to ask is, are there any other true equations of the type in Theorem 1, that is, product of two radii equals product of two other radii?

Figures are at the end of this document.

The first problem is the number of cases to investigate. We can choose four circles out of twelve in ${}^{12}C_4 = 495$ ways, and the four radii can then be arranged two by two on either side of an equation in three ways, so that is $3 \times 495 = 1485$ possible equations to investigate. A crude but effective way to speed things up is to use software to draw the diagram, measure the twelve radii and put the numbers into a spreadsheet, where it is easy to work out the ${}^{12}C_2 = 66$ products, move them into a single column and sort so that any equal pairs come out next to each other. Of the 1485 possible equations, it turns out that just 15 are true, or appear to be true. See Figure 2, where the twelve circles are labelled C_0, C_1, $\ldots$, C_{11}. If the radius of C_i is x_i, all i, then the equations that appear to be true are

$$\begin{array}{ccc}
x_0x_1 = x_2x_3 & x_4x_5 = x_6x_7 & x_8x_9 = x_{10}x_{11} \\
x_4x_{11} = x_8x_7 & x_0x_{10} = x_8x_3 & x_0x_6 = x_4x_2 \\
x_{10}x_5 = x_6x_9 & x_1x_4 = x_6x_3 & x_1x_8 = x_{10}x_2 \\
x_{11}x_1 = x_2x_9 & x_5x_0 = x_2x_7 & x_5x_8 = x_{11}x_6 \\
x_7x_1 = x_3x_5 & x_9x_0 = x_3x_{11} & x_9x_4 = x_7x_{10}
\end{array}$$

The reader is invited to try and spot a pattern here! What is needed is a much better notation, to make the pattern plain, and we shall develop such a notation in Section 3. In the meantime, notice that the fifteen equations are not algebraically independent: for example, the first and fourth ones in the first column together imply the last one in the second column. If we put $y_i = \ln x_i$, all i, then we obtain fifteen *linear* equations in the y_i, and it is an exercise for the reader to write down the corresponding 15×12 matrix of coefficients and check that it has rank 6. So we have, above, six independent equations and nine dependent ones; cf. Proposition 5, below.

2 Preliminaries

We use the standard notation for $\triangle ABC$: $a = |BC|$, $b = |CA|$, $c = |AB|$, and $s = \frac{1}{2}(a + b + c)$. Then the inradius is r, and the exradii opposite A, B, C are r_a, r_b, r_c respectively, and I, I_a, I_b, I_c are the incentre and the excentres opposite A, B, C respectively. By expressing the area Δ as the sum or difference of areas of triangles with base a, b and c in various ways, we obtain the standard results

$$\Delta = rs = r_a(s - a) = r_b(s - b) = r_c(s - c)$$

We also use Heron's formula,

$$\Delta^2 = s(s - a)(s - b)(s - c).$$

From these, we have $rr_a = (s-b)(s-c)$, and also $r_b r_c = s(s-a)$, with similar expressions for rr_b, $r_c r_a$, etc.

Lemma 2. In $\triangle ABC$, suppose the incircle touches BC at P and the excircle opposite A touches BC at Q. Then $rr_a = |BP||PC| = |BQ||QC|$. See Figure 3.

Proof. Immediate from above, since $|BP| = s - b = |QC|$ and $|PC| = s - c = |BQ|$. (An alternative proof uses the fact that $\triangle BPI$ and $\triangle I_aQB$ are similar; details are left to the reader.) ∎

If the reader has not already supplied his or her own proof of Theorem 1, now would be a good time to do so. All that is needed is Lemma 2 together with the fact that the two tangents from a point to a circle are of equal length.

Lemma 3. In $\triangle ABC$, suppose the excircle opposite B touches BC at P and the excircle opposite C touches BC at Q. Then $r_b r_c = |BP||PC| = |BQ||QC|$. See Figure 4.

Proof. Immediate from above since $|BP| = s = |QC|$ and $|PC| = s - a = |BQ|$. (An alternative proof uses the fact that $\triangle BPI_b$ and $\triangle I_cQB$ are similar; details are left to the reader.) ∎

Notice, for future use, that the circles involved in Lemma 2 lie on opposite sides of BC, but on the same side of CA and of AB, whereas the circles involved in Lemma 3 lie on the same side of BC but on opposite sides of CA and of AB.

3 The fifteen theorems

Before we attempt any proofs, we need a good notation, as remarked above. Let the four lines in the diagram be numbered 1, 2, 3, 4, and let T_i be the triangle formed by the three lines *other* than line i, each i. Let the base circle, that touches all four lines, be S.

The other twelve circles will be labelled S_{ij}, where $i, j \in \{1, 2, 3, 4\}$, $i \neq j$; note that there are indeed twelve choices of labels S_{ij}. We now describe how i and j are determined, for any given one of the twelve circles.

Given one of the twelve circles, then, it will touch just three of the four lines, so let line i be the line it does *not* touch, so that our given circle is a touch-circle of T_i, and likewise so is S. So we define j so that *either* our given circle and S lie on opposite sides of line j but on the same side of both line k and of line ℓ, *or else* on the same side of line j but on opposite sides of line k and of line ℓ. See Figure 5.

Now let r be the radius of S, and let r_{ij} be the radius of S_{ij}, all i, j.

As an example, look at the circle C_0 in Figure 2. This circle does not touch line 1, is on the opposite side of line 2 from S, but the same side of line 3 and of line 4, so this circle is S_{12}, and $x_0 = r_{12}$; see Figure 5. As a second example, look at the circle C_5 in Figure 2. This circle does not touch line 3, is on the same side of line 1 as S, but on the opposite side of line 2 and of line 4, so this circle is S_{31}, and $x_5 = r_{31}$; see Figure 5, again. The full list is: $x_0 = r_{12}$, $x_1 = r_{21}$, $x_2 = r_{34}$, $x_3 = r_{43}$, $x_4 = r_{13}$, $x_5 = r_{31}$, $x_6 = r_{24}$, $x_7 = r_{42}$, $x_8 = r_{14}$, $x_9 = r_{41}$, $x_{10} = r_{23}$, $x_{11} = r_{32}$. (We have been careful to avoid confusion of meaning of subscripts here: the single subscripts for x run up as far as 11, but the double subscripts for S and r are never equal to 1 and 0 or to 1 and 1.)

There is an alternative way to determine j. Choose a sense of rotation for the given circle, say clockwise. This in turn imparts a direction to each of the three lines (its tangents) j, k, ℓ. Now look at the direction of rotation these in turn impart by their tangency to S: two will be sending it around one way, and one the opposite way. That odd one out is line j. We leave the reader to check that this gives the same result as before.

Now let the point where line i meets line j be labelled A_{ij}, so that $A_{ij} = A_{j\,i}$. Let the point where S touches line i be labelled P_i, and let $d_{ij} = |A_{ij}P_i|$. Since the tangents from A_{ij} to S are equal, we must have $d_{ij} = d_{j\,i}$. In this notation, we can now combine Lemmas 2 and 3, as follows: if i, j, k, ℓ are distinct, then $rr_{ij} = d_{jk}d_{j\ell}$.

We now have everything we need to state and prove our main result, which, with the notation we have set up, almost proves itself:

THEOREM 4. Let i, j, k, ℓ be a permutation of 1, 2, 3, 4. Then we have

$$\text{(i)}\ \ r_{ij}r_{j\,i} = r_{k\ell}r_{\ell k}$$

$$\text{(ii)}\ \ r_{ik}r_{kj} = r_{i\ell}r_{\ell j}$$

PROOF. (i) We have

$$\begin{aligned} (rr_{ij})(rr_{j\,i}) &= (d_{jk}d_{j\ell})(d_{ik}d_{i\ell}) \\ &= (d_{\ell i}d_{\ell j})(d_{ki}d_{kj}) \\ &= (rr_{k\ell})(r_{\ell k}) \end{aligned}$$

and the result follows on cancelling r^2.

(ii) We have

$$\begin{aligned} (rr_{ik})(rr_{kj}) &= (d_{kj}d_{k\ell})(d_{j\,i}d_{j\ell}) \\ &= (d_{\ell j}d_{\ell k})(d_{j\,i}d_{jk}) \\ &= (rr_{i\ell})(rr_{\ell j}) \end{aligned}$$

and the result follows on cancelling r^2. ∎

Notice that there are three instances of case (i) (of which Theorem 1 is one) and twelve instances of case (ii), so we have our fifteen results. The two cases really are quite different: since S_{ij} is a touch-circle of T_i, each instance of case (i) involves one touch-circle (apart from S) from each of the four triangles, whereas each instance of case (ii) involves two touch-circles (apart from S) from one triangle and one from each of two others. Notice also that each of the twelve circles appears five times in Theorem 4, in one instance of case (i) and in four instances of case (ii).

As remarked earlier, the fifteen equations are not algebraically independent. If we take the case (ii) equation $r_{ik}r_{kj} = r_{i\ell}r_{\ell j}$ and the case (i) equation $r_{\ell j}r_{j\ell} = r_{ik}r_{ki}$, then together they yield $r_{ik}r_{kj}r_{\ell j}r_{j\ell} = r_{i\ell}r_{\ell j}r_{ik}r_{ki}$, or $r_{kj}r_{j\ell} = r_{ki}r_{i\ell}$, which is the case (ii) equation obtained from $r_{ik}r_{kj} = r_{i\ell}r_{\ell j}$ by applying the 4-cycle $(ikj\ell)$ to the subscripts. As a consequence, we have:

Proposition 5. The three case (i) equations

$$r_{12}r_{21} = r_{34}r_{43} \qquad r_{13}r_{31} = r_{24}r_{42} \qquad r_{14}r_{41} = r_{23}r_{32}$$

together with the three case (ii) equations

$$r_{13}r_{32} = r_{14}r_{42} \qquad r_{12}r_{23} = r_{14}r_{43} \qquad r_{12}r_{24} = r_{13}r_{34}$$

algebraically imply the other nine case (ii) equations. ∎

4 Special cases

For a circumscribed or tangential quadrilateral, the four sides of the quadrilateral touch a circle which is *inside* the quadrilateral. But it is also possible for the four sides of a quadrilateral to touch a circle which is *outside* the quadrilateral, as in Figure 6. We propose to call such a quadrilateral *extangential*. (In the tangential case, the base circle is the incircle of two of the triangles T_i, and an excircle of each of the others. In the extangential case, it is an excircle of each of the four triangles.) All fifteen equations in Theorem 4 hold equally well for an extangential quadrilateral: as an example, in Figure 7 we have drawn the circles S (shown dashed), S_{12}, S_{21}, S_{34}, S_{43} in both the tangential case (Theorem 1) and the extangential case, and in both situations the equation $r_{12}r_{21} = r_{34}r_{43}$ holds. However, notice that whereas in the tangential case the circles S_{12} and S_{21} touch opposite sides of the quadrilateral (as do S_{34} and S_{43}), in the extangential case they touch adjacent sides.

Next, suppose lines 1 and 2 (but not lines 3 and 4) are parallel, so that our quadrilateral is a tangential trapezium (but not a parallelogram). Then T_3 and T_4 are no longer proper triangles, as two of their sides are parallel, and the circles S_{31}, S_{32}, S_{41}, S_{42} have disappeared. We are left with S and eight other circles, with $r_{34} = r_{43} = r$. Just five equations survive from Theorem 4, one from case (i), namely

$$r_{12}r_{21} = r^2$$

and four from case (ii), namely

$$r_{12}r_{23} = rr_{14} \qquad rr_{23} = r_{21}r_{14} \qquad r_{12}r_{24} = rr_{13} \qquad rr_{24} = r_{21}r_{13}$$

See Figure 8. There is also an extangential case: see Figure 9.

If lines 1 and 2 are parallel, and also lines 3 and 4, then our quadrilateral is a rhombus, necessarily tangential, and we lose all but five of the circles, namely all but S, S_{12}, S_{21},

S_{34} and S_{43}, all of whose radii are equal, and we then have just one exceedingly trivial equation from Theorem 4. See Figure 10.

As another special case, suppose our quadrilateral is both tangential and extangential, which (assuming there are no parallels) means it is a kite. So we have two choices of base circle: S, inscribed in the kite, and S', touching the four sides produced. Taking S as the base circle, and numbering the lines as in Figure 11, we see that S_{12}, S_{21}, S_{34} and S_{43} all coincide with S'. (To get the numbering right, either take the general case, when line 1 does *not* touch S_{12}, and then take the limit as line 1 is moved until it *does* touch S_{12}; or else just pretend that S' does not touch each of the four lines, in turn.) So now $r_{12} = r_{21} = r_{34} = r_{43} = r'$, the radius of S'. By the symmetry, we also have $r_{23} = r_{14}$, $r_{24} = r_{13}$, $r_{31} = r_{42}$, and $r_{32} = r_{41}$. Denoting these last four quantities by a, b, c and d respectively, we find that four of the equations from Theorem 4 tell us that $ac = bd$, and all the others simply give trivialities, for example $r'a = ar'$.

If, finally, we take the limit of the last case as the meet of lines 1 and 2 goes to infinity, then lines 1 and 2 become parallel, $r = r'$, and we lose four more circles, namely those that touch both line 1 and line 2. We are left with just S_{13}, S_{14}, S_{23} and S_{24}, where $r_{13} = r_{24}$ and $r_{14} = r_{23}$, and now Theorem 4 tells us nothing we did not know already. See Figure 12.

5 Acknowledgement

I originally came across Theorem 1 on Michael De Villiers' website [2], and I am grateful to him for correspondence in which he encouraged me to write up these results.

6 References

1. Fukagawa, H. & Rigby, J. (2002). Traditional Japanese Mathematics Problems of the 18th and 19th Centuries. Singapore: SCT Publishing, p. 22.

2. http://dynamicmathematicslearning.com/japanese-circum-quad-theorem.html

John R. Silvester *14 March 2017*
Department of Mathematics, King's College
Strand, London WC2R 2LS
(Email: jrs@kcl.ac.uk)

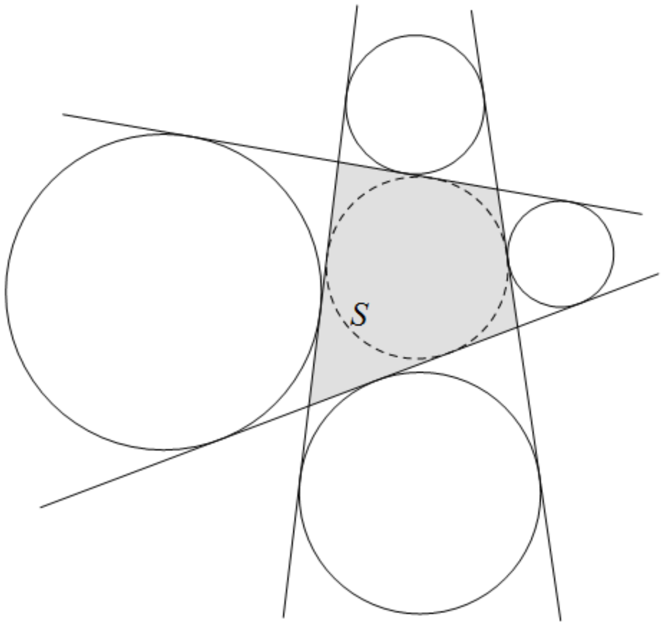

Figure 1

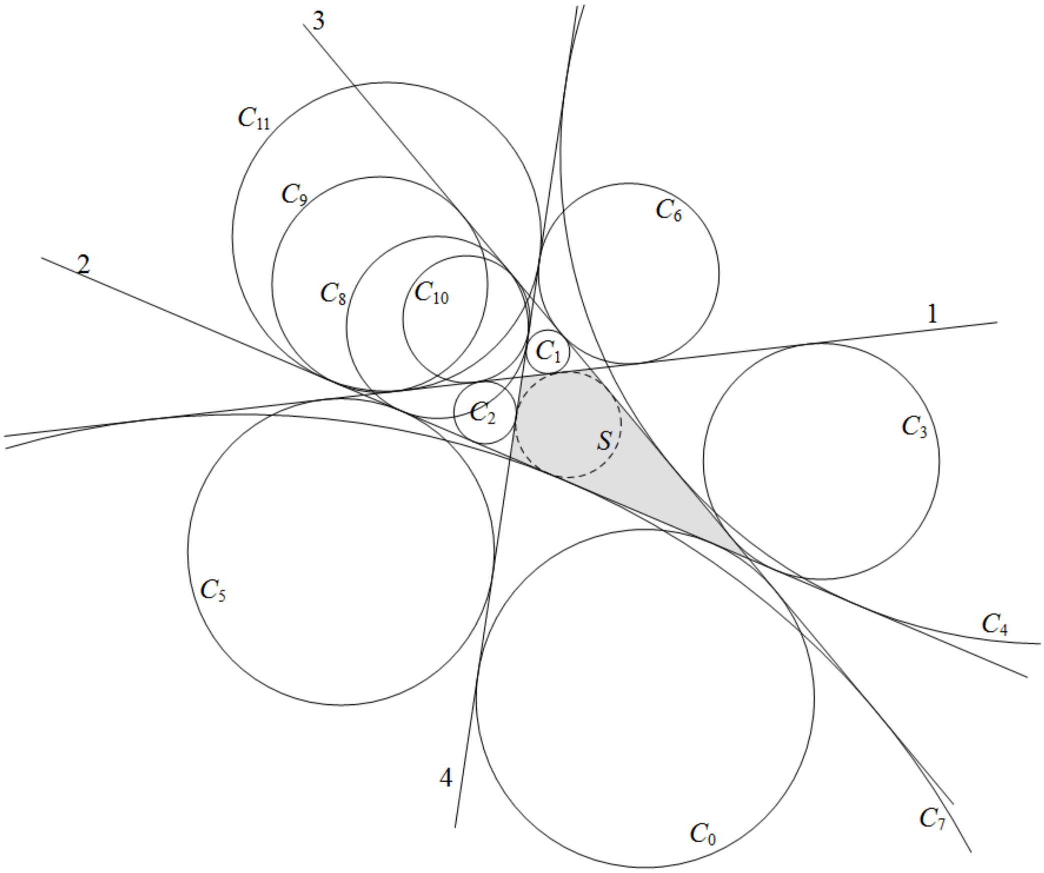

Figure 2

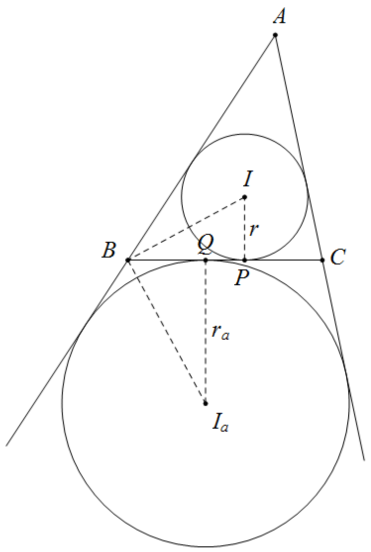

Figure 3

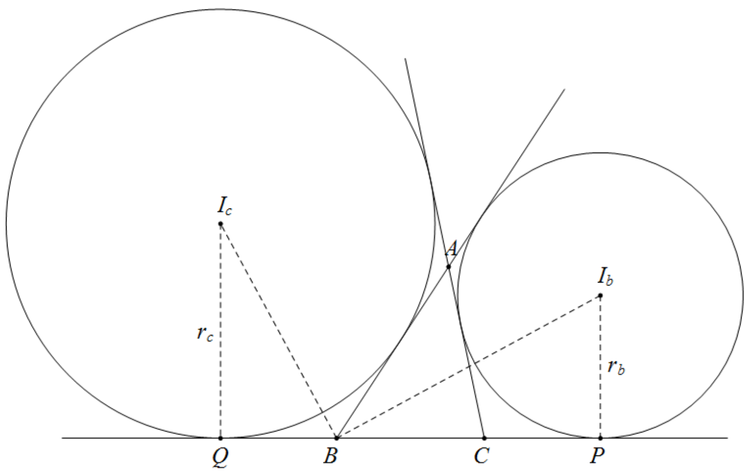

Figure 4

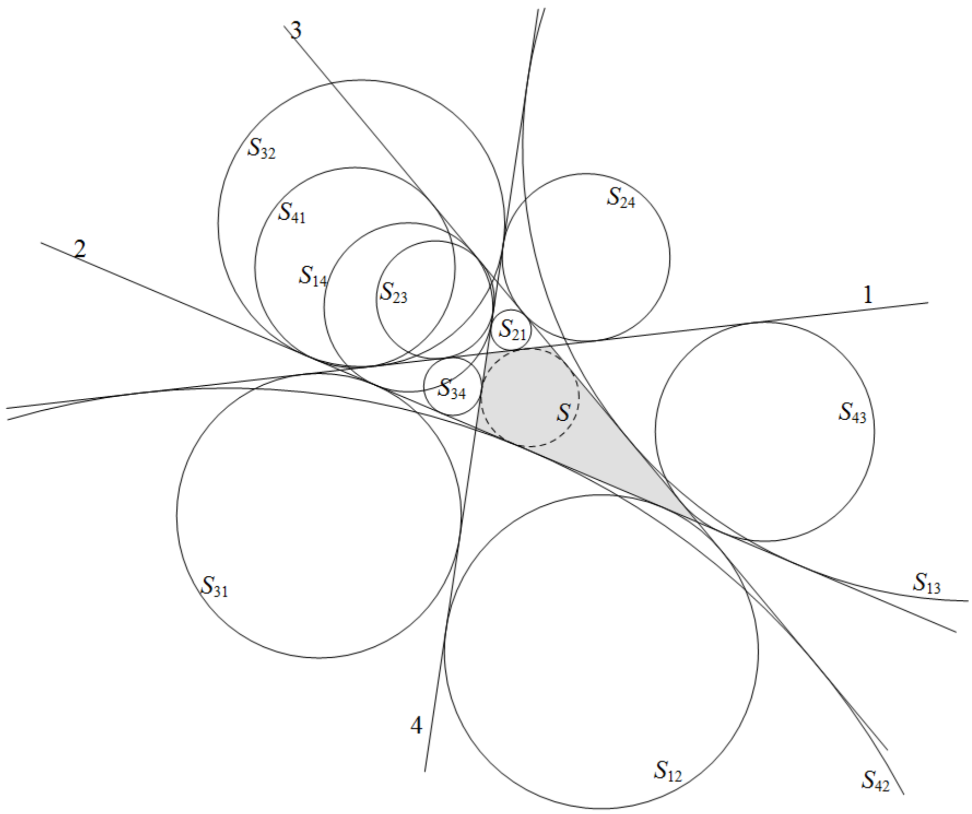

Figure 5

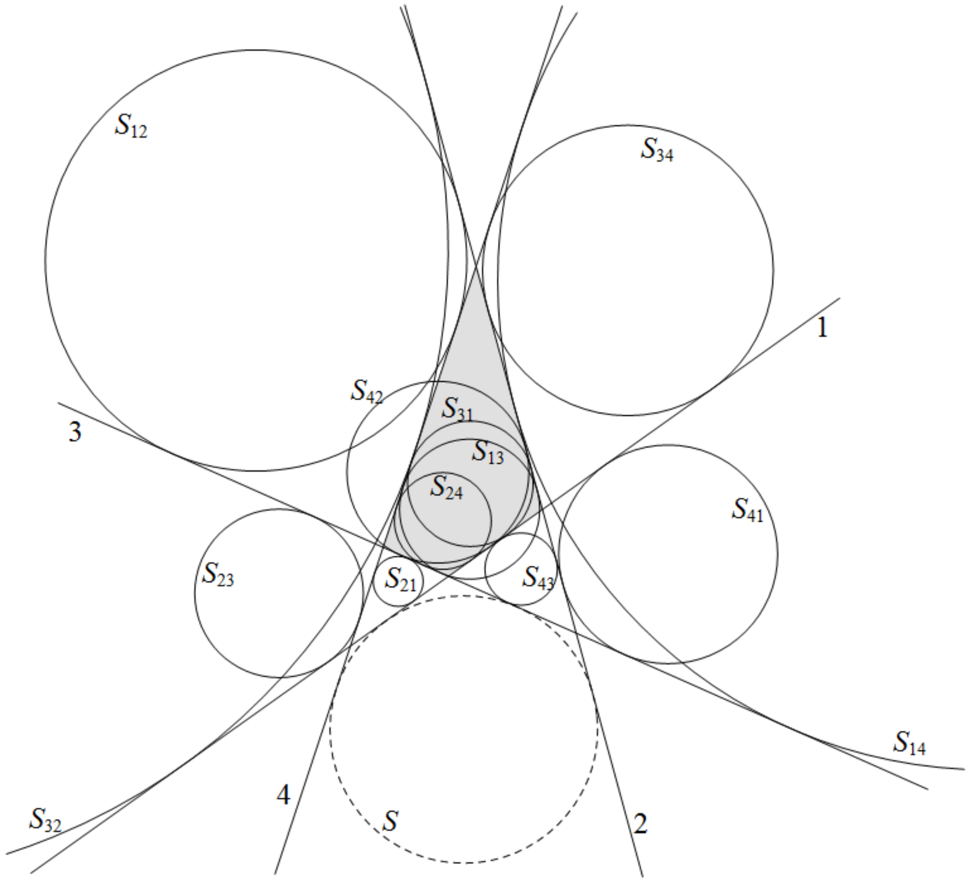

Figure 6

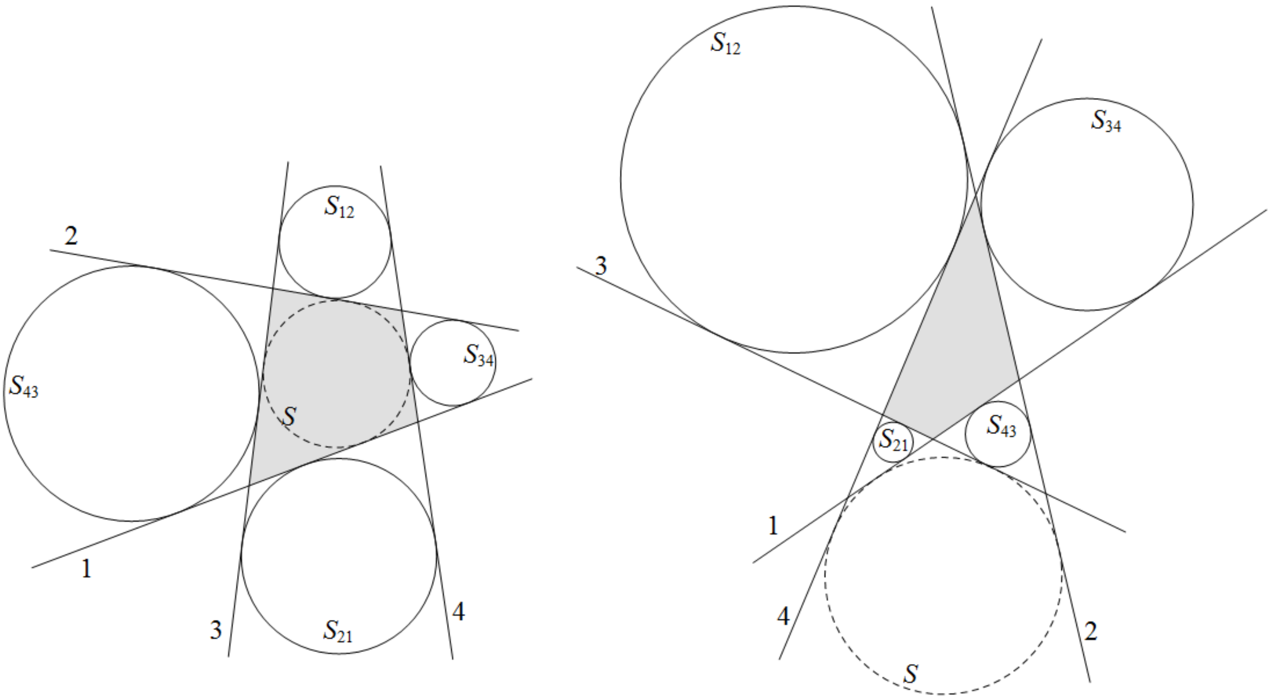

Figure 7

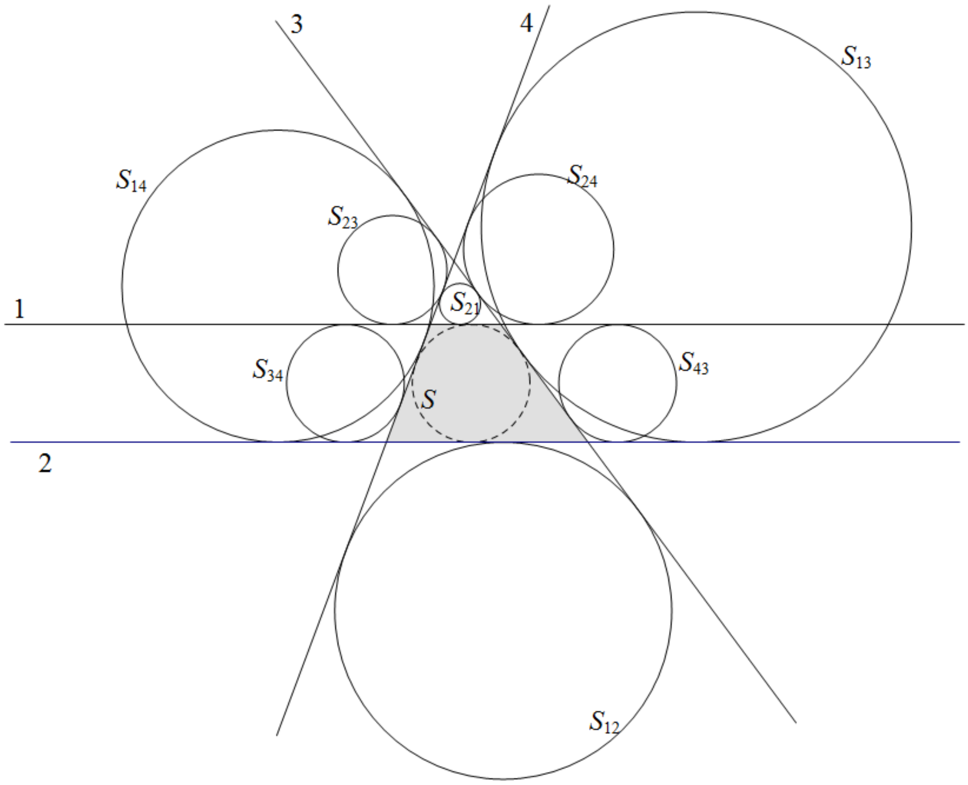

Figure 8

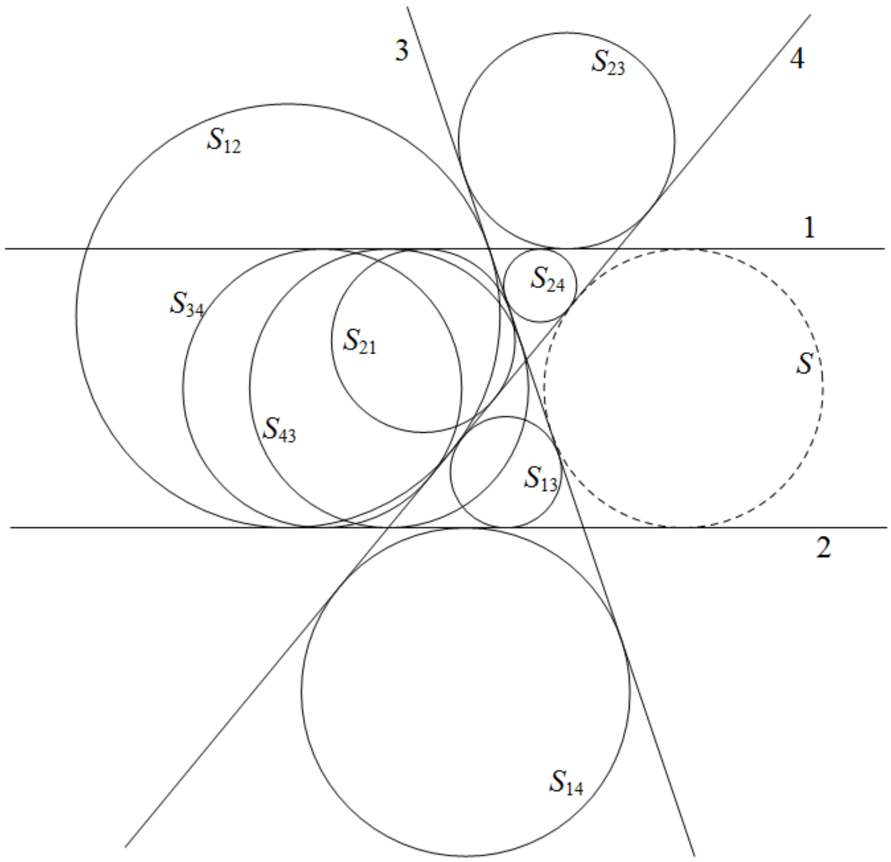

Figure 9

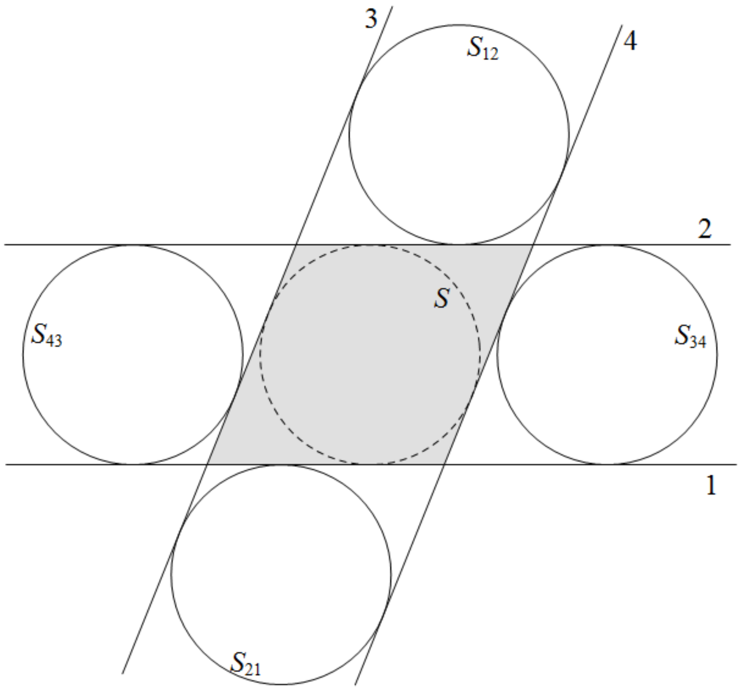

Figure 10

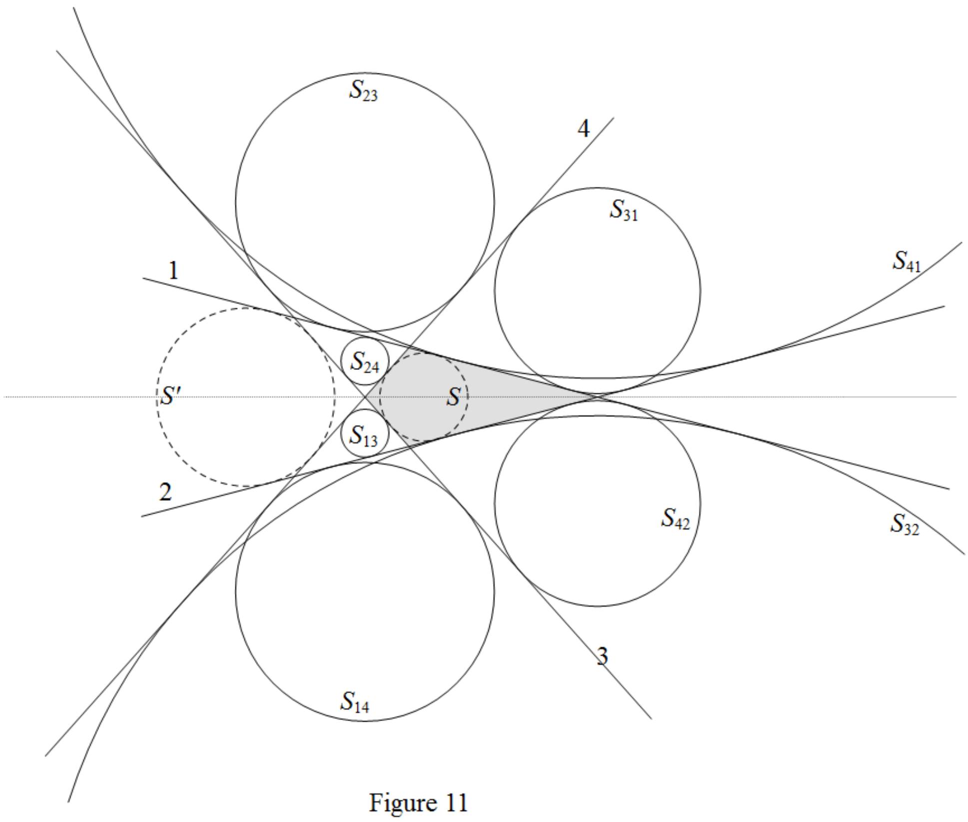

Figure 11

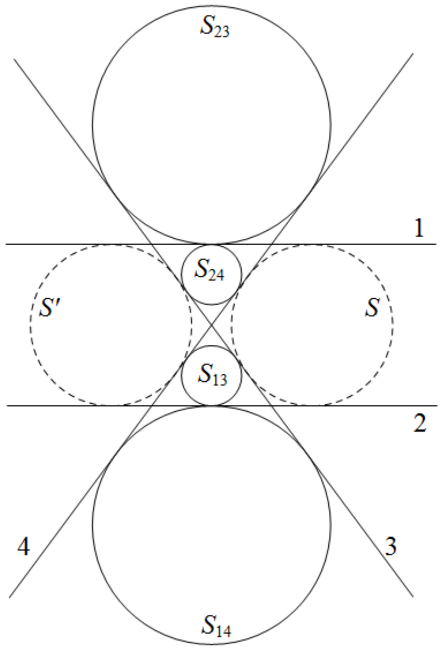

Figure 12