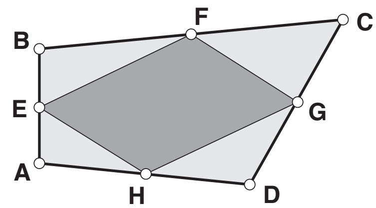
Varignon Area

In this activity, you will compare the area of a quadrilateral to the area of another quadrilateral constructed inside it.

CONJECTURE



Open the sketch **Varignon Area.gsp** and drag vertices to investigate the shapes in this sketch.

1. Points *E*, *F*, *G*, and *H* are midpoints of the sides of quadrilateral *ABCD*. Describe polygon *EFGH*.

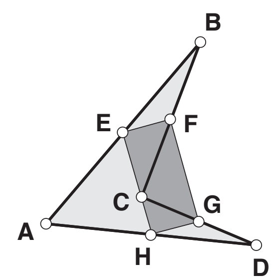
Press the appropriate button to show the areas of the two polygons you described. Drag a vertex and observe the areas.

2. Describe how the areas are related. You might want to find their ratio.

To find the ratio between two measurements, choose **Calculate** from the Number menu and then click on a measurement to enter it into the Calculator.

3. Drag any of the points *A*, *B*, *C*, and *D* and observe the two area measurements. Does the ratio between them change?

4. Drag a vertex of *ABCD* until it is concave. Does this change the ratio of the areas?



5. Write your discoveries so far as one or more conjectures. Use complete sentences.

6. You probably can think of times when something that always appeared to be true turned out to be false sometimes. (The previous activity, Areas, was a geometric example of this kind of occurrence.) How certain are you that your conjecture is always true? Record your level of certainty on the number line and explain your choice.

CHALLENGE If you believe your conjecture is always true, provide some examples to support your view and try to convince your partner or members of your group. Even better, support your conjecture with a logical explanation or a convincing proof. If you suspect your conjecture is not always true, try to supply counterexamples.

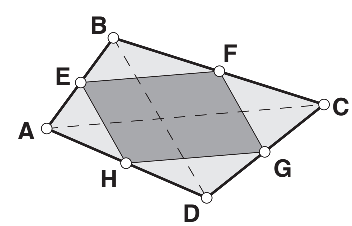
PROVING

In the picture, you probably observed that quadrilateral *EFGH* is a parallelogram. You also probably made a conjecture that goes something like this:

> The area of the parallelogram formed by connecting the midpoints of the sides of a quadrilateral is half the area of the quadrilateral.

This first conjecture about quadrilateral *EFGH* matches a theorem of geometry that is sometimes called Varignon's theorem. Pierre Varignon was a priest and mathematician born in 1654 in Caen, France. He is known for his work with calculus and mechanics, including discoveries that relate fluid flow and water clocks.

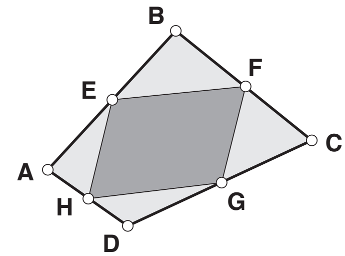
The next three steps will help you verify that quadrilateral *EFGH* is a parallelogram. If you have verified this before, skip to Question 10.



7. Construct diagonal $\overline{AC}$. How are $\overline{EF}$ and $\overline{HG}$ related to $\overline{AC}$? Why?

8. Construct diagonal *BD*. How are $\overline{EH}$ and $\overline{FG}$ related to $\overline{BD}$? Why?

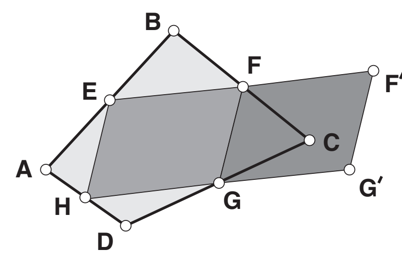
9. Use Questions 7 and 8 to explain why *EFGH* must be a parallelogram.



Work through the steps that follow for one possible explanation as to why parallelogram *EFGH* has half the area of quadrilateral *ABCD*. (If you have constructed diagonals in *ABCD*, it will help to delete or hide them.)

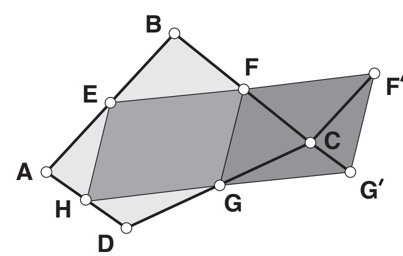
10. Assume for now that *ABCD* is convex. One way to explain why *ABCD* has twice the area of *EFGH* is to look at the regions that are inside *ABCD* but not inside *EFGH*. Describe these regions.

11. According to your conjecture, how should the total area of the regions you described in Question 10 compare with the area of *EFGH*?



Press the button to translate the midpoint quadrilateral *EFGH* along vector *EF*.

12. Drag any point. How does the area of the translated quadrilateral compare to the area of *EFGH*?



Construct $\overline{F'C}$ and $\overline{G'C}$.

13. How is ΔEBF related to $\Delta F'CF$?

14. Explain why the relationship you described in Question 13 must be true.

15. How is ΔHDG related to $\Delta G'CG$?

16. Explain why the relationship you described in Question 15 must be true.

17. How is ΔAEH related to $\Delta CF'G'$?

18. Explain why the relationship you described in Question 17 must be true.

19. You have one more triangle to account for. Explain how this last triangle fits into your explanation.

Present Your Proof

Create a summary of your proof from Questions 10–19. Your summary may be in paper form or electronic form and may include a presentation sketch in Sketchpad. You may want to discuss the summary with your partner or group.

Further Exploration

Which part of your proof does not work for concave quadrilaterals? Try to redo the proof so that it explains the concave case as well. (*Hint:* Drag point *C* until quadrilateral *ABCD* is concave.)

also shown that this ratio is precisely 5 when the midvexogram is a trapezoid and that in all other cases the ratio is always less than 6 (although there are quadrilaterals for which this ratio can be as close to 6 as wanted). Their paper can be downloaded from http://www.ms.uky.edu/~carl/coleman/coleman2.html.

This proof, though convincing, is hardly explanatory, and the problem of finding a short, elegant, and explanatory geometric proof remains open.

VARIGNON AREA (PAGE 76)

This activity follows the Areas activity, and it is expected that students will be a bit more skeptical here about their Sketchpad observations and thus more motivated to seek additional verification or conviction. The focus of this activity is therefore on introducing the verification function of proof.

Prerequisites: The Kite Midpoints activity or knowledge of the result that the line connecting the midpoints of two sides of a triangle is parallel to the third side and half its length. Properties of parallelograms. Conditions for congruency.

Sketch: Varignon Area.gsp.

CONJECTURE

1. *EFGH* is a parallelogram. (This is true even for concave and crossed cases.)
2. The area of the parallelogram is half that of the original quadrilateral.
3. No.
4. No.
5. The midpoints of the sides of a quadrilateral form a parallelogram.
6. Responses will vary.

PROVING

7. $\overline{EF} \parallel \overline{AC} \parallel \overline{HG}$, since *E* and *F* are midpoints of sides *AB* and *CB* in triangle *ABC* and *H* and *G* are midpoints of sides *AD* and *CD* in triangle *ADC*.
8. $\overline{EH} \parallel \overline{BD} \parallel \overline{FG}$ (same reasons).
9. $\overline{EF} \parallel \overline{HG}$ and $\overline{EH} \parallel \overline{FG}$, so opposite sides are parallel, and therefore *EFGH* is a parallelogram. Another way of proving it is to note in Question 7 that not only is $\overline{EF} \parallel \overline{HG}$, but since both *EF* and *HG* are equal to half *AC,* they are also equal to each other. So one pair of opposite sides are equal and parallel, from which it follows that *EFGH* is a parallelogram.

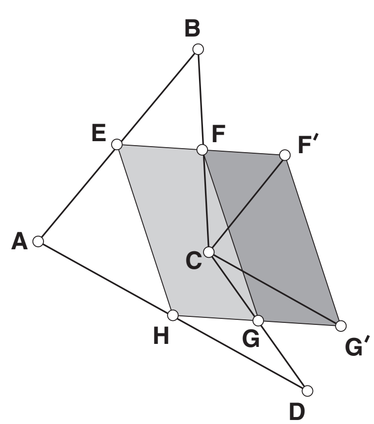
   Note: You may also wish to ask your students to prove that the result is also true in the concave and crossed cases. The proofs are similar, except that now one or both diagonals fall outside.

10. There are four triangles lying outside *EFGH*, namely, *AEH*, *DHG*, *CGF*, and *BFE*.
11. The sum of the areas of these triangles must be equal to the area of *EFGH*.
12. The translated quadrilateral is congruent to *EFGH* (property of translation), so it is also a parallelogram with area equal to that of *EFGH*.
13. ΔEBF is congruent to $\Delta F'CF$ (SAS).
14. $FB = FC$ (F is midpoint of BC); $FE = FF'$ (corresponding sides of translated parallelograms), and $m\angle EFB = m\angle F'FC$ (directly opposite angles).
15. ΔHDG is congruent to $\Delta G'CG$ (SAS).
16. Similar to Question 14.
17. ΔAEH is congruent to $\Delta CF'G'$ (SSS).
18. From Question 13, we have $CF' = BE$ and $BE = AE$. Therefore, $AE = CF'$. Similarly, from Question 17, we have $AH = CG'$. Also, $EH = FG$ (corresponding sides of translated parallelograms).
19. *FGC* is common to both *ABCD* and *FGG′F′*. Therefore, the sum of the areas of the triangles is equal to that of *FGG′F′*, and therefore to that of *EFGH*.

Present Your Proof

This section provides an opportunity for students to synthesize the argument and write it up in a coherent way.

Further Exploration



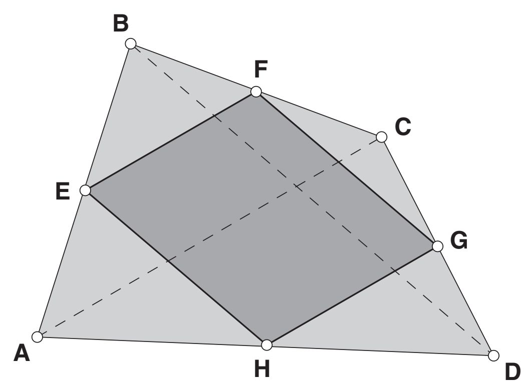
This proof is a little tricky. In the concave case, only three triangles, namely *AEH*, *DHG*, and *BFE*, fall within *ABCD*. The remaining triangle *CGF* now falls outside *ABCD*. If we use the notation (XYZ) to represent the area of a polygon *XYZ*, then $(ABCD) = (AEH) + (BFE) + (DHG) + (EFGH) - (CGF) = (AEH) + (BFE) + (DHG) - (CGF) + (EFGH)$. In other words, we now have to prove that $(AEH) + (BFE) + (DHG) - (CGF) = (EFGH)$. From the translation, *EFGH* is still congruent to *FGG′F′*. As before, triangles *EBF* and *F′CF*, triangles *HDG* and *G′CG*, and triangles *AEH* and *CF′G′* are congruent. But if we subtract the area of triangle *CGF* from the sum of the areas of triangles *G′CG*, *CF′G′*, and *F′CF*, we obtain the area of parallelogram *FGG′F′*, which is equal to that of *EFGH*.

Alternative Proof

There are several different ways of proving this result. It might be instructive for your students to work through hints such as those given here.

Hints

1. Express the area of *EFGH* in terms of the area of *ABCD* and the areas of triangles *AEH*, *CFG*, *BEF*, and *DHG*.



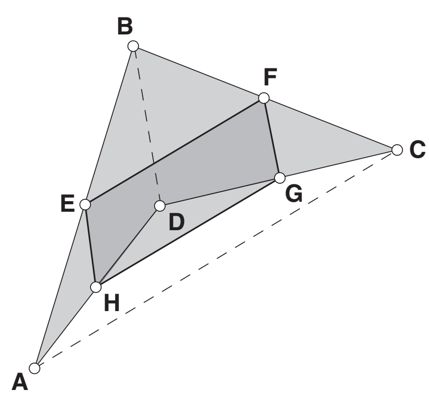
2. Drop a perpendicular from *A* to $\overline{BD}$ and express the area of triangle *AEH* in terms of the area of triangle *ABD*.
3. Similarly, express the areas of triangles *CFG*, *BEF*, and *DHG*, respectively, in terms of the areas of *CBD*, *BAC*, and *DAC*, and substitute in step 1.
4. Simplify the equation in step 3 to obtain the desired result.

Proof

1. Using the notation (XYZ) for the area of a polygon XYZ, we have $(EFGH) = (ABCD) - (AEH) - (CFG) - (BEF) - (DHG)$.
2. If the height of ΔABD is h, then $(ABD) = \frac{1}{2}BD \cdot h$ and $(AEH) = \frac{1}{2}\left(\frac{1}{2}BD\right) \cdot \frac{1}{2}(h) = \frac{1}{4}(ABD)$, or simply, the base and the height are half those of the large triangle.
3. $(EFGH) = (ABCD) - \frac{1}{4}(ABD) - \frac{1}{4}(CBD) - \frac{1}{4}(BAC) - \frac{1}{4}(DAC)$.
4. $(EFGH) = (ABCD) - \frac{1}{4}(ABCD) - \frac{1}{4}(ABCD)$
$= \frac{1}{2}(ABCD)$.

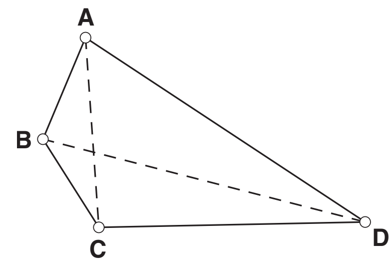
Further Discussion

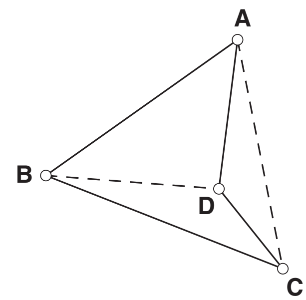
You may also want your students to work through an explanation for the concave case, because it is generically different. For example, unless the notation is carefully reformulated (e.g., see crossed quadrilaterals below), the equation in step 1 of the proof does not hold in the concave case, but becomes $(EFGH) = (ABCD) - (AEH) - (CFG) - (BEF) + (DHG)$ (see below). However, substituting into this equation as before, and simplifying, leads to the same conclusion.



Crossed Quadrilaterals

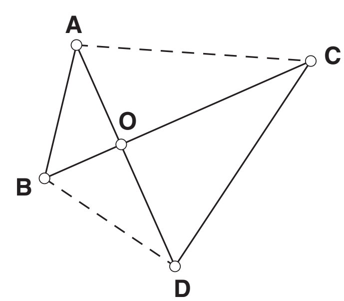
It is also true for the crossed quadrilateral *ABCD* that *EFGH* has half its area, as some of your students may have found on Sketchpad. However, the proof is even more tricky and first requires consideration of what we mean by the area of a crossed quadrilateral. Let us now first carefully try to define a general area formula for convex and concave quadrilaterals. It seems natural to define the area of a convex quadrilateral to be the sum of the areas of the two triangles into which it is decomposed by a diagonal. For example, diagonal $\overline{AC}$ decomposes the area as follows (see first figure): $(ABCD) = (ABC) + (CDA)$.





In order to make this formula work for the concave case as well (see second figure), we obviously need to define $(CDA) = -(ADC)$. In other words, we can regard the area of a triangle as being *positive* or *negative* depending on whether its vertices are named in *counterclockwise* or *clockwise* order. For example:

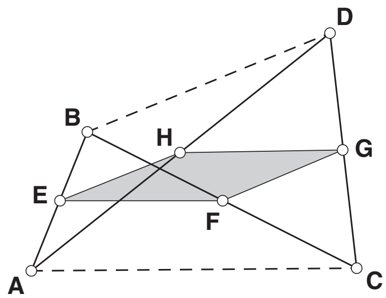
$$(ABC) = (BCA) = (CAB) = -(CBA) = -(BAC) = -(ACB)$$



Applying the above formula and definition of area in a crossed quadrilateral (see figure), we find that diagonal *AC* decomposes its area as follows:

$$(ABCD) = (ABC) + (CDA) = (ABC) - (ADC)$$

In other words, this formula forces us to regard the "area" of a crossed quadrilateral as the difference between the areas of the two small triangles *ABO* and *ODC*. (Note that diagonal *BD* similarly decomposes $(ABCD)$ into $(BCD) + (DAB) = -(DCB) + (DAB)$). An interesting consequence of this is that a crossed quadrilateral will have zero "area" if the areas of triangles *ABO* and *ODC* are equal.



Using this valuable notation, the result can now simultaneously be proved for all three cases (convex, concave, and crossed) as follows:

$$(EFGH) = (ABCD) - (AEH) - (FCG) - (EBF) - (DHG)$$

$$= (ABCD) - \frac{1}{4}(ABD) - \frac{1}{4}(CDB) - \frac{1}{4}(BCA) - \frac{1}{4}(DAC)$$

$$= (ABCD) - \frac{1}{4}(ABCD) - \frac{1}{4}(ABCD)$$

$$= \frac{1}{2}(ABCD)$$

LOGICAL PARADOX (PAGE 80)

This worksheet is based on an example that has often been used (wrongly) to try to motivate a need for proof among students. Basically, students are told that this example illustrates that diagrams may be deceiving and therefore unreliable. Consequently, reliance only on experimental evidence is unreliable and we thus require formal proof.

However, this example actually illustrates the importance of making (reasonably) *accurate* diagrams when constructing proofs, rather than showing that diagrams are unreliable. In fact, the false conclusion that all triangles are isosceles shows how easily a correct logical argument can lead to a fallacy because of a construction error, or a mistaken assumption, in a sketch. Instead of motivating a need for proof, such examples actually emphasize the importance of experimental testing (i.e., the *accurate* construction of some examples), noting with care the relative positions of points, lines, and so on that are essential to the proof. Although a French mathematician once said "Geometry is the art of drawing correct conclusions from incorrectly drawn sketches," this example dramatically shows that they should not be constructed too incorrectly!

Prerequisites: Knowledge of conditions for congruency.

Sketch: Paradox.gsp (This sketch should be given to students only at the end of the worksheet, after they have worked through the logical argument based on the faulty diagram.)

CONJECTURE

1. Triangles *CGD* and *CGF* are congruent (SAA).
2. $DG = FG$.
3. $AG = BG$, since *G* lies on the perpendicular bisector of $\overline{AB}$.
4. Triangles *GDA* and *GFB* are congruent (90°, S, S).
5. $DA = FB$.
6. $CD = CF$.
7. $CD + DA = CA = CF + FB = CB$.
8. Therefore, *ABC* is isosceles.