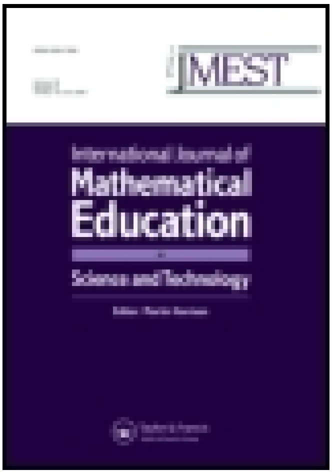

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Vertical line and point symmetries of differentiable functions

Michael D. De Villiers [a]
[a] Faculty of Education , University of Durban-Westville , Durban 4000, South Africa
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Vertical line and point symmetries of differentiable functions

by MICHAEL D. DE VILLIERS
Faculty of Education, University of Durban-Westville,
Durban 4000, South Africa

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A heuristic account is given of the author's personal investigation of some aspects of the vertical line and point symmetries of differentiable functions. Starting from examples of third degree polynomials there is first generalized to polynomials in general, and later even more generally to differentiable functions. The role of quasi-empirical testing is demonstrated throughout, not only in the gaining of confidence in conjectures, but also in improving them through the production of counter-examples. In some examples the role of deductive proof is also clearly shown to be far less that of verification than that of explanation, systematization and/or discovery.

Symmetry, as wide or as narrow as you may define it, is one idea by which man through the ages has tried to comprehend, and create order, beauty and perfection. (Hermann Weyl).

1. Introduction

Unfortunately in reporting the results of their investigations mathematicians invariably use the so-called 'deductivist style'. This is regrettable mainly for two reasons, the one more philosophical and the other more didactical. Firstly it sheds no light on the actual process of mathematical discovery and suggests to the uninformed that new discoveries in mathematics are always made by purely deductive reasoning. Secondly, this style in the words of Lakatos ([1], p. 102) 'hides the struggle, hides the adventure . . .' and 'an authoritarian air is secured for the subject by beginning with disguised monster-barring and proof-generated definitions and with the fully fledged theorem, and by suppressing the primitive conjecture, the refutations, and the criticism of the proof'.

In contrast to the deductivist style stands the 'heuristic style' which does not one-sidedly focus only on the finished end-product, but tries to illuminate the gradual evolution of mathematical ideas from their ancestral primitives. The heuristic approach therefore usually discusses among other things, the original motivating problem-situation, the interplay between intuitive and formal reasoning, false generalizations, incomplete proofs and finally the significance and utility of the discovered or invented results. Critics of this approach, however, usually say that mathematical papers and text-books would then be far too long and comprehensive to read to the end. But as Lakatos ([1], p. 144) has said: 'The answer to this pedestrian argument is: let us try.' In presenting this paper I make a modest attempt at the heuristic style in preference to the deductivist style.

2. The original problem

Symmetry as we know abounds in nature, e.g. in the snowflake, the arrangement of the petals of a flower and the leaf of a tree. It is often also employed by artists to create visually pleasing configurations. Similarly, I have always found it visually pleasing and intellectually satisfying to discover graphs with line symmetry and/or points of symmetry and to investigate their properties. What follows is the distillation of some personal ideas on the symmetries of differentiable functions.

Traditional textbooks for first year college mathematics like Allendoerfer and Oakley [2] and Ayres [3] basically discuss the symmetries of graphs only in relation to the coordinate axes or other lines through the origin (reflective symmetry) and the origin itself (point symmetry). For instance, a function $y=f(x)$ is defined to be symmetric around the y-axis if and only if $y=f(x) \Leftrightarrow y=f(-x)$ and point symmetric in relation to the origin if and only if $y=f(x) \Leftrightarrow y=-f(-x)$. (Note that if a graph is point symmetric in relation to the origin, it is invariant under a half-turn (a rotation of 180°) around the origin.) In relation to a graph of a cubic polynomial such as the one shown in Figure 1, traditional textbooks therefore either make no mention of any symmetry (e.g. [3], p. 98) or explicitly state 'There is no symmetry...' (e.g. [2], p. 394).

3. Solution

However, looking at the graph of the function shown in Figure 1, I intuitively visualized a point of symmetry approximately between $x=1$ and $x=2$. How could I find this point of symmetry and prove that it was one? After some consideration I hypothesized that if the graph was point symmetric with regard to a certain point, its gradient to the left of that point must be the same as the gradient to the right of that point; i.e. reflective symmetric. By roughly drawing the derivative $dy/dx = 3x^2 - 8x - 3$ on the same axes I then realized that the point of symmetry must lie on the line of symmetry of the derivative. The substitution of the vertical line of symmetry for the derivative, namely $x = 4/3$ into the original function, thus gave me a hypothesized point of symmetry, namely (4/3; 250/27). To prove that this was

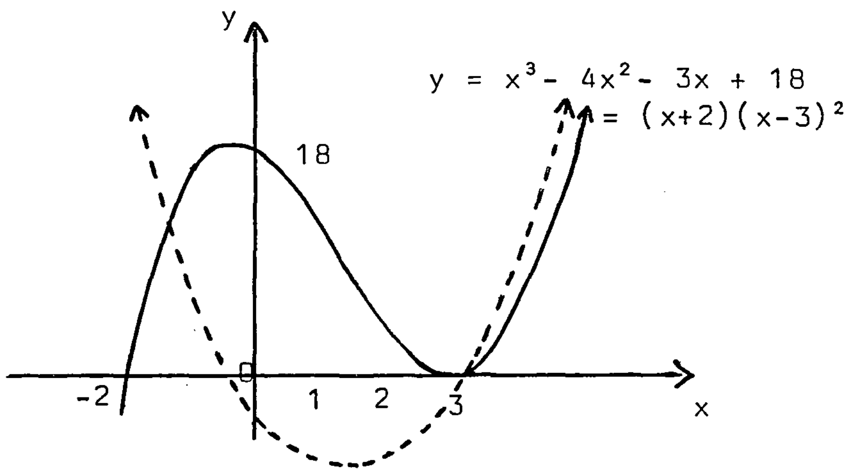

Figure 1.

indeed a point of symmetry one needed only to translate the function to the origin and test for point symmetry in relation to the origin. For example, the transformed equation is:

$$y=(x+4/3+2)(x+4/3-3)^2-250/27$$

which simplifies to:

$$y=x^3-(25/3)x$$

However, this equation is equivalent to $y=-((-x)^3-(25/3)(-x))$ and therefore proves that the point (4/3; 250/27) is a point of symmetry of the graph of the original equation.

4. Other problems and generalizations

After considering some more cubic polynomials and investigating them for possible points of symmetry, I eventually hypothesized that 'all third degree polynomials are point symmetric around a point $(a; f(a))$ where $x=a$ is the vertical line of symmetry of the derivative'. (Obviously the fact that all parabola are reflective symmetric, also strongly encouraged this hypothesis.) A proof of this result is given in De Villiers [4].

The next step was to investigate some quartic polynomials for vertical lines of symmetry, for example, graphs like $y=x^4-x^2$ as shown in Figure 2. In this case the graph is symmetric around the y-axis ($x=0$), while the derivative $dy/dx=4x^3-2x$ is point symmetric around the origin (0; 0). Furthermore, translations of the graph of $y=x^4-x^2$ to various positions in the plane shifts its vertical line of symmetry, but the derivative remains point symmetric with its point of symmetry moving with the vertical line of symmetry of the original function. At this point, I therefore made the bold intuitive leap to the following two generalizations:

(1) If a polynomial function is reflective symmetric, then its derivative is point symmetric.

(2) If a polynomial function is point symmetric, then its derivative is reflective symmetric.

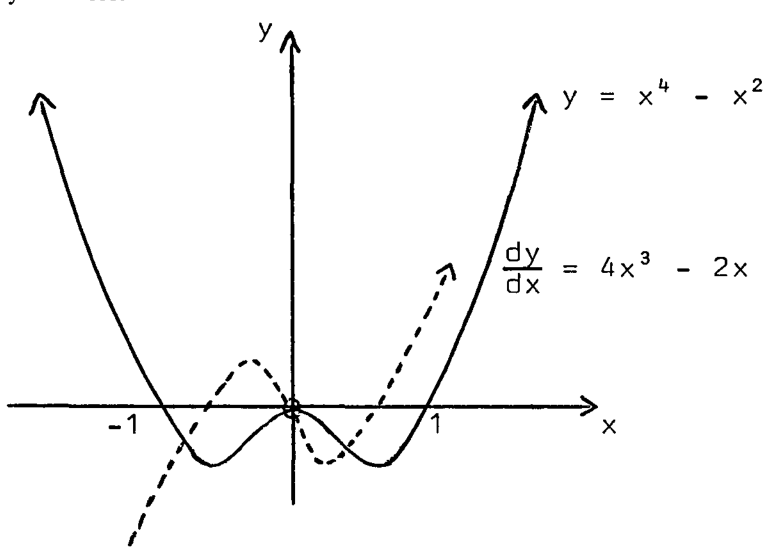

Figure 2.

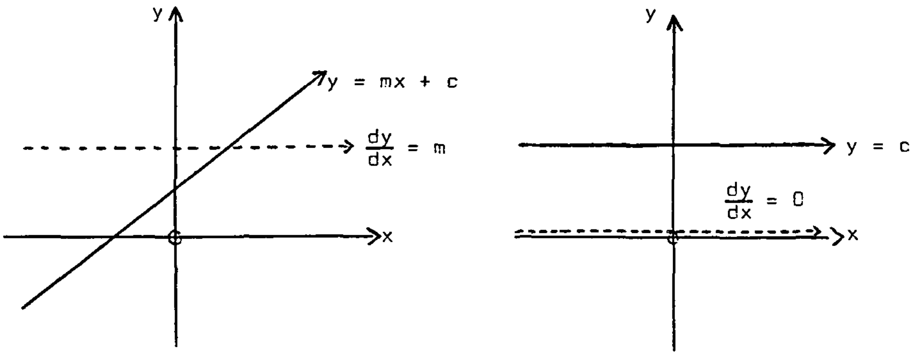

Figure 3.

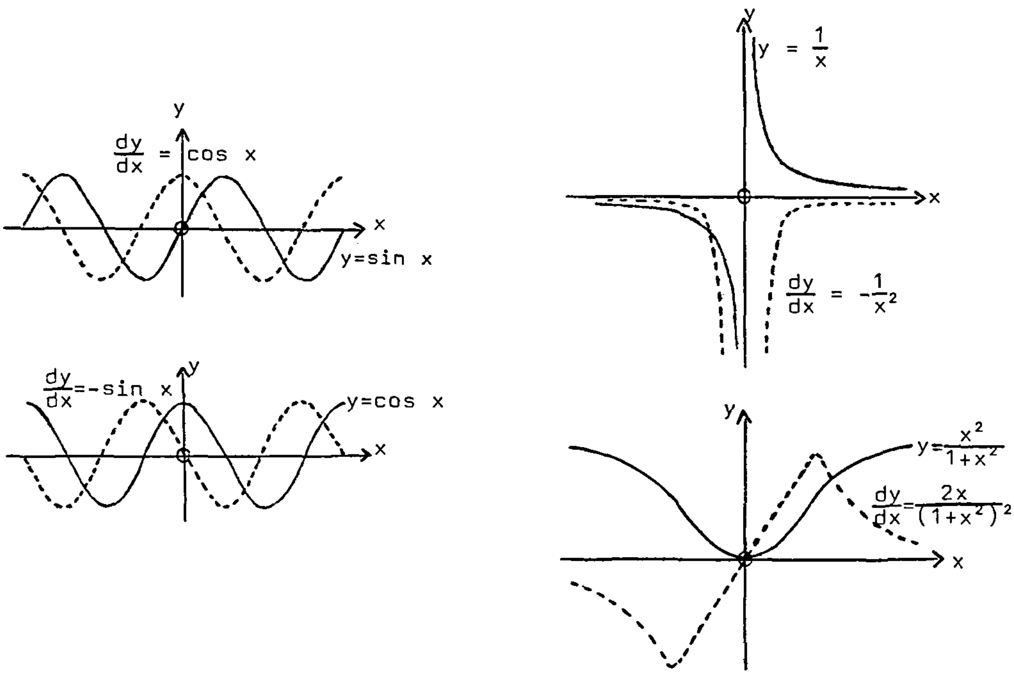

Figure 4.

5. Special cases

To quickly check these two conjectures I next considered the two special cases shown in Figure 3. In the first case, the line $y=mx+c$ has an infinite number of points of symmetry, namely all the points on the line, while its derivative $\mathrm{d}y/\mathrm{d}x=m$ on the other hand, has an infinite number of vertical lines of symmetry (as well as an infinite number of points of symmetry). In the second case the line $y=c$ has an infinite number of vertical lines of symmetry as well as an infinite number of points of symmetry while the derivative also has both an infinite number of points of symmetry and an infinite number of lines of vertical symmetry. The second case thus confirms both conjectures at the same time!

6. Extension and checking

What about functions other than polynomials? Or are these conjectures restricted to polynomial functions only? Proceeding to check these conjectures with the graphs of some other functions such as those shown in Figure 4, I soon realized that it did not seem to be restricted to the polynomial functions alone, but seemed to apply to all differentiable functions. To convince myself with regard to their validity for any composite function, I proceeded to check these conjectures for the following two cases:

(1) $y = \tan x = \sin x / \cos x$ (point symmetry at (0; 0))

$dx/dy = 1/\cos^2 x$ (symmetric around $x = 0$)

(2) $y = (9 - x^2)^{1/2}$ (symmetric around $x = 0$)

$dy/dx = -x/(9 - x^2)^{1/2}$ (point symmetry at (0; 0)).

7. Counter-example and further generalization

Although the converse of the second conjecture also seemed valid in all cases, I soon realized that the converse of the first conjecture (as stated above) was not necessarily valid. For example, if we consider the graph of $y = x^4 - x$ as shown in Figure 5, we find that it does not have vertical line symmetry, although its derivative $dy/dx = 4x^3 - 1$ has a point of symmetry at (0; -1). Note that the first case in Figure 3 also provides another counter-example where the derivative $dy/dx = m$ has an infinite number of points of symmetry, but $y = mx + c$ has no vertical line of symmetry. For a function to be symmetric around a vertical line, it is therefore a necessary but not

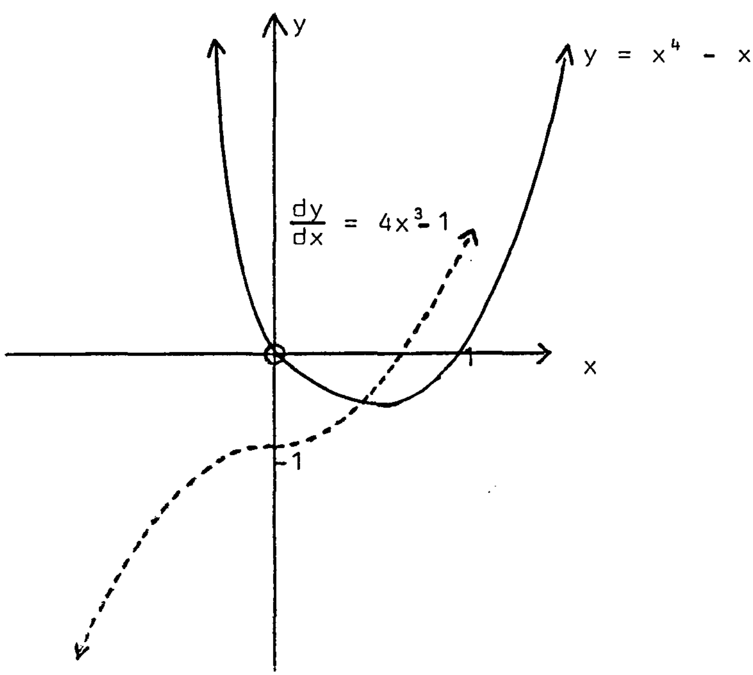

Figure 5.

sufficient condition that its derivative is point symmetric: it is a necessary and sufficient condition if and only if the point of symmetry of the derivative lies on the x-axis. I consequently reformulated my two earlier conjectures as follows:

(1) A differentiable function $y=f(x)$ is reflective symmetric around a vertical line $x=a$ if and only if its derivative $dy/dx=f'(x)$ is point symmetric around the point $(a;0)$.
(2) A differentiable function $y=f(x)$ is point symmetric around a point $(a;b)$ if and only if its derivative $dy/dx=f'(x)$ is reflective symmetric around the vertical line $x=a$.

What is especially appealing about these two conjectures above are that they exhibit an elegant duality between reflective symmetry and point symmetry, and thus also between vertical line and point. They also have the following two useful corollaries, namely:

(3) A differentiable function is reflective symmetric around a vertical line $x=a$ if and only if its second derivative is reflective symmetric around the same line $x=a$ (and the first derivative is point symmetric at $(a;0)$).
(4) A differentiable function is point symmetric around a point $(a;b)$ if and only if its second derivative is point symmetric around the point $(a;0)$.

It should also be emphasized at this point that with a differentiable function is not meant here a function that is necessarily differentiable for all x (and therefore continuous $\forall x$), but includes functions like $y=1/x$ (see second figure in Figure 4) not differentiable at certain values of x (e.g. $x=0$). From my experience with polynomial functions, I then intuitively sensed that if a point symmetric function was known to be differentiable $\forall x$, its point of symmetry had to lie on the function itself. Thus, if the function referred to in conjecture (2) above was differentiable (and therefore continuous) $\forall x$, it would be point symmetric specifically around $(a; f(a))$, and not just some general point $(a;b)$.

8. Intuitive geometric proofs

But how could one explain conjectures (1) and (2)? Although a great amount of quasi-empirical testing may provide one with a lot of confidence, it cannot however by itself, provide insight into why the conjectures are true. Geometrically they seem quite obvious when one considers the two cases shown in Figure 6, in each case for both a continuous and a discontinuous graph. In the first case, if $y=f(x)$ is reflective symmetric around $x=a$ it means that the graph to the left of $x=a$ must fit exactly on the graph to the right of $x=a$, and vice versa. Therefore the y-values of the graphs respectively to the left and right of $x=a$ are exactly equal in sign and size, and local minima and maxima on the left correspond exactly with those on the right. However the gradient (derivative) of the graph to the left of $x=a$ is opposite in sign to the gradient (derivative) to the right of $x=a$, although equal in size. For example, the derivative of the graph on the left of $x=a$ is negative when the derivative of the corresponding part of the graph to the right of $x=a$ is positive, and vice versa. The same applies for example to the local maxima and minima of the derivatives to the left and right of $x=a$, where local maxima on the left (e.g. at A) corresponds exactly to local minima at the right (e.g. at A′), and vice versa. However, these are the

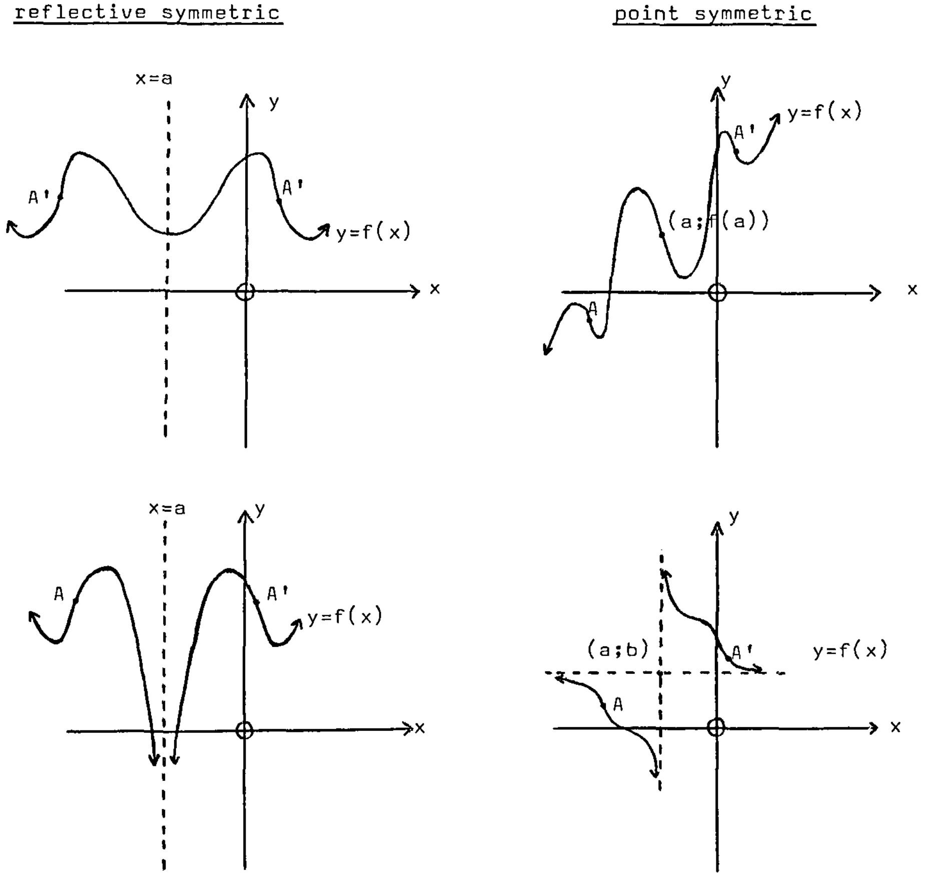

Figure 6.

properties of a graph which is point symmetric at $(a; 0)$ (see second case below), and it therefore means that the derivative in the first case is point symmetric at $(a; 0)$.

Similarly, in the second case if $y=f(x)$ is point symmetric with regard to $(a; b)$, the graph to the left of $x=a$ can be made to fit exactly on the graph to right of $x=a$ by a half-turn (a rotation through 180°), and vice versa. Therefore, local minima to the left of $x=a$ correspond exactly to local maxima to the right of $x=a$, and vice versa, but the y-values of the graph to the left and right of $x=a$ are exactly equal in size and opposite in sign only if the point of symmetry lies on the x-axis (e.g. at $(a; 0)$). (See first case above.) Let's now consider the gradient (derivative) of the function $y=f(x)$ as given in the second case. Clearly in this case the gradient (derivative) of the graph to the left of $x=a$ is exactly equal to the gradient (derivative) to the right of $x=a$, not only in size but also in sign. Note that local minima and maxima of the derivative on the left correspond exactly with those on the right. For example, the local minimum on the left at A corresponds exactly with the local minimum on the right at A′. Since these are the properties of a graph which is reflective symmetric around $x=a$ (see first case above), it means that the derivative in the second case is reflective symmetric around $x=a$.

The proofs of the converses are now quite simple: by integrating $dy/dx=f'(x)$ in either case, we obtain $y=f(x)+c$, which means that it is simply a vertical translation

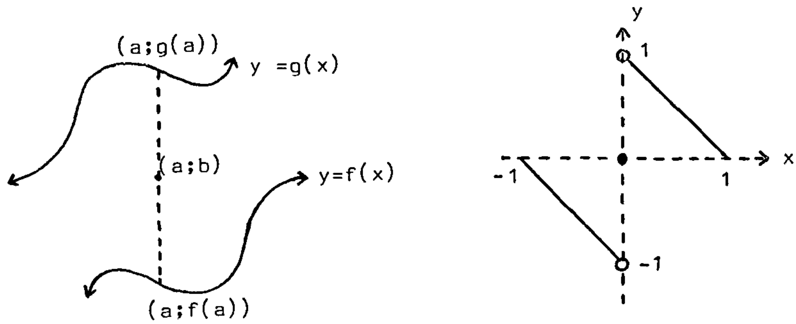

Figure 7.

of the original graph of $y=f(x)$ with a value of c. However, this translation clearly does not affect the symmetric properties of the original graph and therefore completes the proof in both cases.

To set about proving the conjecture that the point of symmetry of a point symmetric function $y=f(x)$ which is differentiable $\forall x$, lies on the function itself, I used the first sketch shown in Figure 7. Suppose its point of symmetry, namely $(a; b)$, did not lie on the function $y=f(x)$ itself, but somewhere outside the function as shown. Then a rotation of $y=f(x)$ through 180° around the point $(a; b)$ gives us the image $y=g(x)$. But according to the definition of point symmetry, this image $y=g(x)$ must coincide with the original function, which means that the original function must have consisted of two separate parts $y=f(x)$ and $y=g(x)$. This is however clearly a contradiction, since we would then have had two different images $y=f(a)$ and $y=g(a)$ for $x=a$, and then we would not have had a function to begin with! Thus, we conclude that the point of symmetry $(a; b)$ must lie on the function $y=f(x)$ itself.

Reflecting upon this proof and looking at further examples, I soon realized with then obviously seeming hindsight, that the differentiability and/or continuity of the point symmetric function were sufficient conditions, but not necessary conditions for the point of symmetry to lie on the function itself, In fact, the whole proof hinges only around the property that $y=f(x)$ is uniquely defined at $x=a$. Thus, it is directly applicable to, for instance, a point symmetric function defined as $x=0$, $y=-x+1$ for $1 \geqslant x > 0$ and $y=-x-1$ for $-1 \leqslant x < 0$ (see second figure in Figure 7). This result was then generally reformulated as follows:

> If a function $y=f(x)$ has a point of symmetry at $(a; b)$ and it is defined at $x=a$, then $(a; b)$ lies on the function itself.

Now clearly the converse of this result is also trivially true. For instance, if the point of symmetry $(a; b)$ of a function lies on the function itself, it obviously follows that the function must therefore be defined at $x=a$. Furthermore, since the differentiability of a function at $x=a$ implies continuity at that point, which in turn also implies that the function is defined at $x=a$, the aforementioned conditions are clearly special conditions of the general result formulated above. However, the converses of both these special cases are false (a counter-example for both cases is given by the second figure in Figure 7).

9. Formal analytical proofs

Although I was by now sufficiently convinced of the general validity of these three conjectures by the production of the above arguments (in combination with the preceding experimental investigations), I still felt a need to prove them analytically in terms of the standard transformation equations. Personally, the function of the following proofs was therefore not to eliminate any doubts, but simply to systematize these results neatly in relation to the familiar conceptual techniques of analysis and transformation geometry. (Compare de Villiers [5].) (It will also be noted by the reader that the geometric proofs convey a greater sense of explanation than the analytical proofs below.)

Theorem 1. A differentiable function $y=f(x)$ is reflective symmetric around a vertical line $x=a$ if and only if its derivative $dy/dx=f'(x)$ is point symmetric around the point $(a;0)$.

Proof. Consider a differentiable function $y=f(x)$ which is reflective symmetric around $x=a$. Translate the graph of the function horizontally to lie symmetric around the y-axis. Therefore $y=f(x+a)\Leftrightarrow y=f(-x+a)$.

Now consider the derivative of the left side of the above equivalence:

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+a+h)-f(x+a)}{h} \tag{1}$$

The right side of the above equivalence is, however, a composite function. To determine its derivative we set $u=-x$ and first determine dy/du as follows:

$$\frac{dy}{du}=\lim_{h\to 0}\frac{f(u+a+h)-f(u+a)-}{h}$$

But $du/dx=-1$, and therefore:

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$$=\lim_{h\to 0}\frac{-[f(-x+a+h)-f(-x+a)]}{h} \tag{2}$$

Since $y=f(x+a)$ and $y=f(-x+a)$ are equivalent forms of the same function, the derivatives of both forms must clearly also be equivalent. Therefore, (1)$\Leftrightarrow$(2), which implies that the derivative dy/dx is point symmetric in relation to the origin $(0;0)$. If we now translate the graph $y=f(x+a)$ back to its original position by moving it horizontally by a units, the derivative is clearly translated by the same amount so that its point of symmetry moves to $(a;0)$. This then completes the proof of the first part of the first conjecture. The converse can now be proved in the same way as shown earlier by starting from the equivalence between (1) and (2).

Theorem 2. A differentiable function $y=f(x)$ is point symmetric around a point $(a;b)$ if and only if its derivative $dy/dx=f'(x)$ is reflective symmetric around the vertical line $x=a$.

Proof. Consider a differentiable function $y=f(x)$ which is point symmetric around $(a; b)$. Translate the graph of the function so that it lies point symmetric around the origin $(0; 0)$. Therefore, $y=f(x+a)-b \Leftrightarrow y=-f(-x+a)+b$. Now consider the derivative of the left side of the above equivalence:

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+a+h)-f(x+a)}{h} \tag{3}$$

Since the right side of the equivalence is also a composite function we find its derivative as in the preceding proof:

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(-x+a+h)-f(-x+a)}{h} \tag{4}$$

Similarly to the previous proof, the two forms of the derivative above are also equivalent, e.g. (3)$\Leftrightarrow$(4). Therefore the derivative dy/dx is reflective symmetric around the y-axis. A translation of the graph of $y=f(x+a)-b$ vertically by b units clearly does not affect the point symmetry at $(0; 0)$ of its derivative (the derivative of a constant is 0). Therefore, a translation back to its original position will only involve a horizontal translation of the derivative by a units, which means that the vertical line of symmetry has moved from the y-axis to $x=a$. This then completes the proof of the first part of the second conjecture. The converse can now be proved in the same way as shown earlier on.

Theorem 3. A point symmetric function has a point of symmetry $(a; b)$ lying on the function itself, if and only if the function is defined at $x=a$.

Proof. If $(a; b)$ lies on the function, it follows directly that $b=f(a)$ and therefore that the function is defined at $x=a$. This then completes the first part of the proof. To prove the converse without *reductio ad absurdum*, we can proceed as follows. If a function $y=f(x)$ is point symmetric at $(a; b)$, then

$$y=f(x+a)-b \Leftrightarrow y=-f(-x+a)+b$$

(see previous proof) or equivalently

$$f(x+a)-b=-f(-x+a)+b \tag{5}$$

However, if $y=f(x)$ is defined for $x=a$, then $y=f(x+a)-b$ will be defined at $x=0$ (the original graph was horizontally translated by a units). Therefore, we may substitute $x=0$ in equation (5) to obtain:

$$f(a)-b=-f(a)+b \Leftrightarrow f(a)=b$$

Which means that $(a; b)$ lies on the function itself, and completes the proof of the converse.

10. Further reformulation

Although having proved the first two conjectures as theorems, I proceeded to further explore them in a quasi-empirical fashion (a lá Lakatos) by considering a great variety of cases, some of which are shown in Figure 8. Note that I here only explored the first theorem, as similar dual examples could be constructed for the second theorem. From the first two examples shown, it quickly became clear that a function with a 'corner' on its vertical line of symmetry is merely analogous to the second reflective symmetric case considered in Figure 6, where the function is discontinuous on the vertical line of symmetry and the derivative also does not exist. However, such 'corners' (or discontinuities) need not lie on the vertical line of symmetry as shown by the third case in Figure 8, but need only lie symmetrical with respect to the vertical line of symmetry.

Consider now the fourth case shown in Figure 8. Clearly the derivative ($dy/dx=1$ for $x>2$ and $dy/dx=-1$ for $x<2$) of this reflective symmetric function is point symmetric around (0; 0), thus exemplifying the theorem. However, if one shifted the point (0; 2) on the reflective symmetric function to say (−1; 2) its reflective symmetry is destroyed, but its derivative remains point symmetric around (0; 0). The point symmetric property of the derivative in this case, was therefore not a sufficient condition for its integral to be reflective symmetric. So here we shockingly have a counter-example to the converse part of Theorem 1. Disaster! But how could this be? Did I not formally prove Theorems 1 and 2?

My initial reaction was of course in defence of both my theorems; in other words, I simply thought that this counter-example fell beyond their domain of validity.

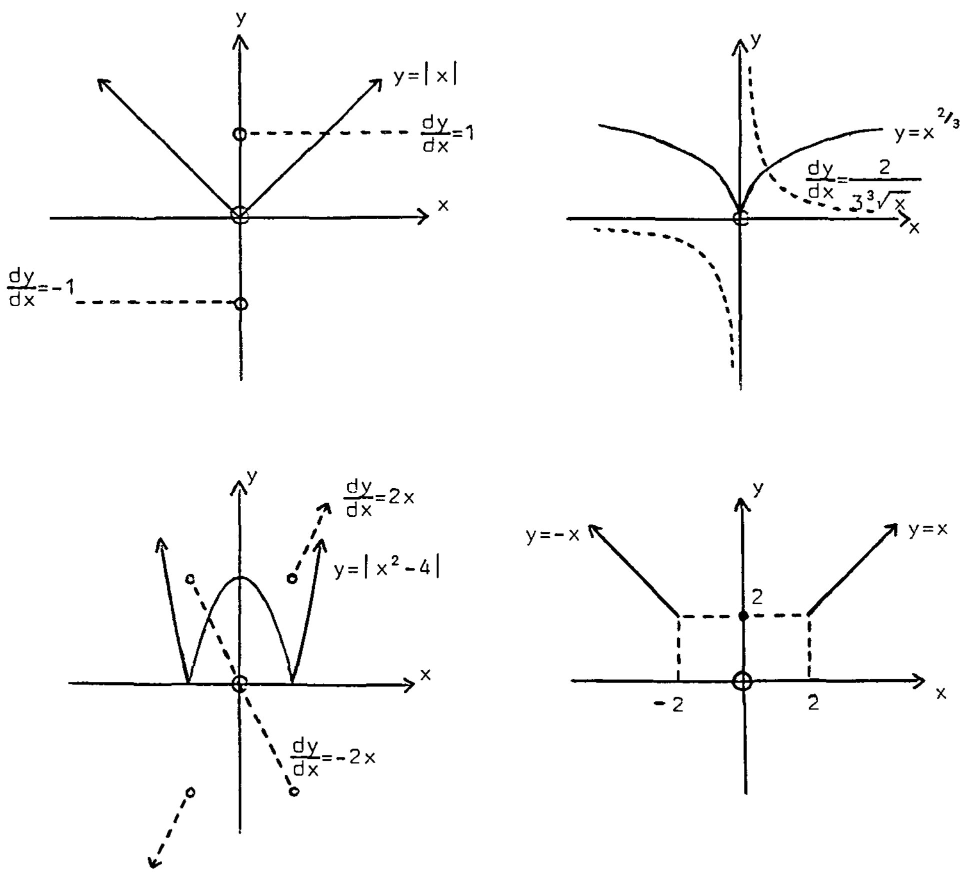

Figure 8.

(Compare with the process of 'monster barring' described by Lakatos, ([1], p. 42–47). However proceeding to recheck my proofs and more closely examine the domain of validity of the theorems in question, the problem was then found to lie completely with a subconscious restriction of the actual domain of validity. The fundamental aspect of both theorems was not as I had subconsciously thought, that the function as a whole should be symmetrical, but only that the differentiable parts of the function should be symmetrical! This was clearly already evident (with hindsight) in the formal proofs, as they obviously could only refer to those parts of the function which are differentiable. Without affecting the validity of the proofs, I therefore extended the first two theorems by simply reformulating them as follows.

Theorem 1. The differentiable parts of a function $y=f(x)$ are reflective symmetric around a vertical line $x=a$ if and only if the derivative(s) $dy/dx=f'(x)$ of these parts are point symmetric around the point $(a;0)$.

Theorem 2. The differentiable parts of a function $y=f(x)$ are point symmetric around a point $(a;b)$ if and only if the derivative(s) $dy/dx=f'(x)$ of these parts are reflective symmetric around the vertical line $x=a$.

It should furthermore be noted here that had I not embarked on further quasi-empirical examination of the theorems in question, I would probably not have discovered the above pathological case leading to their further extension. This example therefore confirms the danger of a blind reliance on deductive proof as the final and only authority in mathematical research. According to Lakatos ([1], p. 143) such a distorted reliance 'is the worst enemy of independent and critical thought'. (Also compare [5].)

Finally, since $y=1/x$ is point symmetric with respect to the origin we would expect its integral to be reflective symmetric around the y-axis. The usual definition, however, for this integral is

$$\int 1/x \, dx = \ln x$$

with $x>0$, which is not reflective symmetric around a vertical axis. In order to maintain the symmetry of these two theorems we would therefore have to define

$$\int 1/x \, dx = \ln |x|$$

with $x \in R$ giving us a natural logarithm graph which lies symmetric with respect to the y-axis.

11. Utility and application

Clearly the first two theorems embody a certain sense of beauty and harmony, as well as a remarkable simplicity. They are mysterious on the one hand, but also intuitively obvious on the other hand. It is for instance still with a sense of wonderment and excitement that I look at the well-known exotic graphs of symmetric functions such as shown in Figure 9 and realize that their derivatives will be correspondingly symmetric, whatever they may be!

The utility of these two theorems and their corollaries obviously lie in their application to finding possible lines of symmetry or points of symmetry in the graphs of especially polynomial functions. For instance, let us suppose we wanted to find out

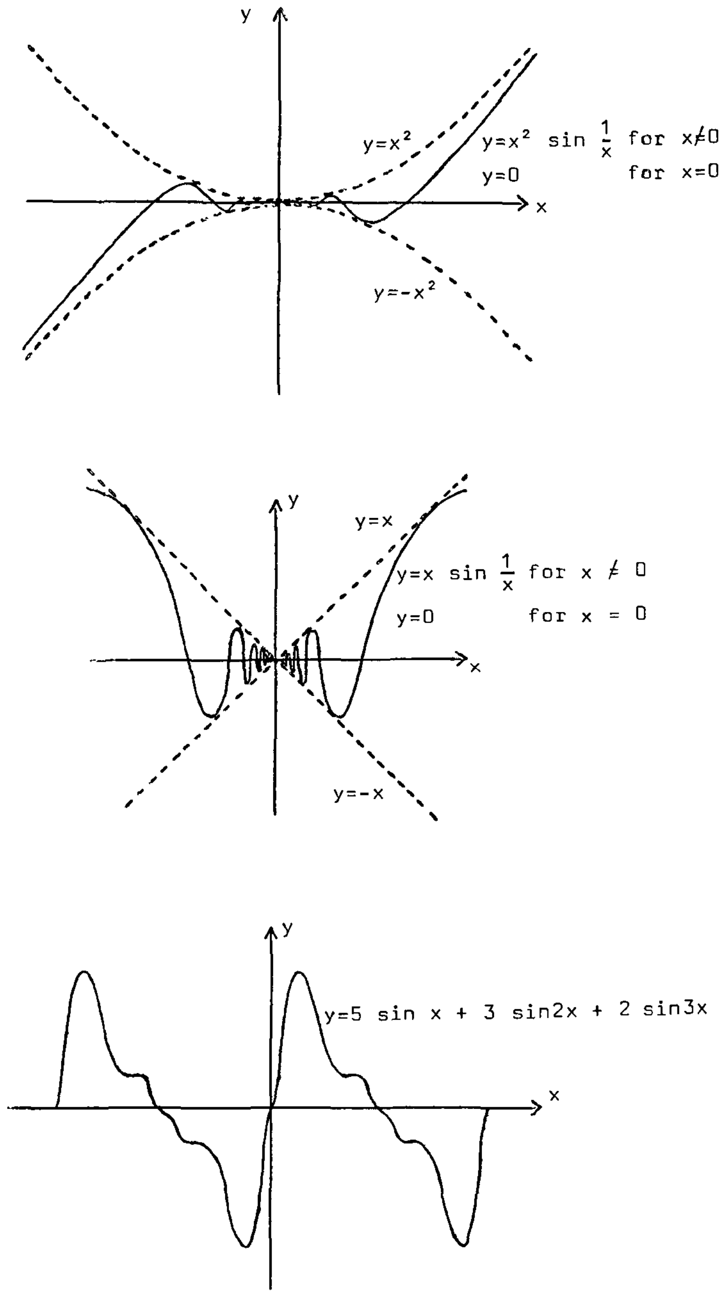
y
y=x²
y=x² sin 1/x for x≠0
y=0 for x=0
x
y=-x²
y
y=x
y=x sin 1/x for x ≠ 0
y=0 for x = 0
x
y=-x
y
y=5 sin x + 3 sin2x + 2 sin3x
x

Figure 9.

if the polynomial $y=f(x)=x^4-9x^2+7x+4$ has vertical line symmetry. We then only need to determine the first derivative $f'(x)=4x^3-18x+7$ and then the second derivative $f''(x)=12x^2-18$. The second derivative is clearly symmetric around the y-axis ($x=0$). Therefore, according to Theorems 2 and 3, the point of symmetry of the first derivative will lie at (0; 7). But according to Theorem 1 this is not a sufficient condition for $y=x^4-9x^2+7x+4$ to have a vertical line of symmetry at $x=0$, since it is necessary that the point of symmetry of the first derivative lies on the x-axis. We therefore conclude that this polynomial has no vertical line symmetry.

For another example let's suppose we want to find out if the polynomial function $y=f(x)=x^5/5+x^4+(5/3)x^3+x^2$ has a point of symmetry. We can then find the first, second and third derivatives as follows:

$$f'(x)=x^4+4x^3+5x^2+2x$$

$$f''(x)=4x^3+12x^2+10x+2$$

$$f'''(x)=12x^2+24x+10$$

The third derivative is clearly reflective symmetric around $x=-24/(2\times 12)=-1$, which means through substitution into the second derivative according to Theorems 2 and 3, that the second derivative is point symmetric at $(-1; 0)$. However, according to Corollary 2 this is a sufficient condition to conclude that the original function is point symmetric, and if we solve $f(-1)$, this point is given by $(-1; 2/15)$.

12. Further problems and new conjectures

These theorems are also useful in finding the points of symmetry of functions other than polynomials or proving that they have none. Consider, for example, the rational function defined by

$$f(x)=\frac{x^2+1}{x-3}$$

and its derivative given by

$$f'(x)=\frac{x^2-6x-1}{(x-3)^2}$$

When I considered the derivative $f'(x)$ I found that both the numerator and the denominator were reflective symmetric around $x=3$. I then intuitively felt that this had to mean that the derivative itself was reflective symmetric around $x=3$. This would then in turn mean according to Theorem 2 that $f(x)$ was point symmetric somewhere along the line $x=3$! To check if the derivative was indeed reflective symmetric, I excitedly translated it by 3 units to the left, e.g.

$$f'(x+3)=\frac{(x+3)^2-6(x+3)-1}{(x+3-3)^2}$$

$$=\frac{x^2-5}{x^2}$$

Since each x can clearly be replaced by $-x$ without affecting the equation, $f'(x+3)$ is clearly reflective symmetric around $x=0$, and therefore $f'(x)$ is reflective symmetric around $x=3$ as suspected.

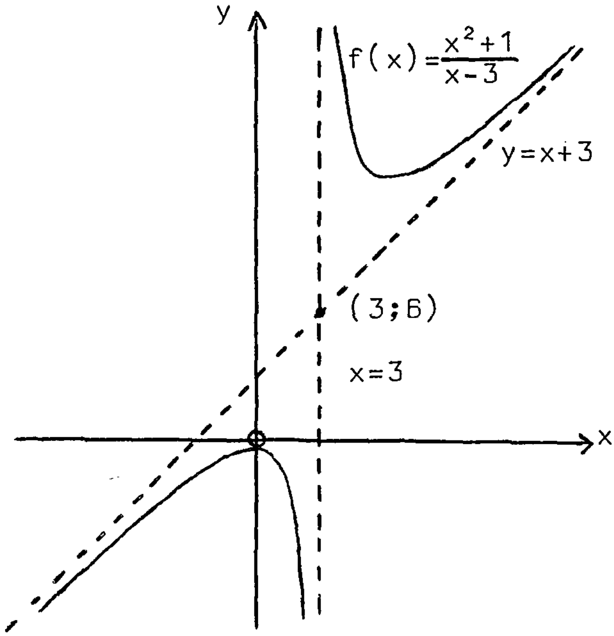

Figure 10.

Since the function is undefined for $x=3$, the point of symmetry cannot lie on the function itself (Theorem 3), but somewhere along the line $x=3$ which in this case is an asymptote. The question was, however, where on this line was it to be found? Noticing that the function also has a skew asymptote of $y=x+3$, I immediately suspected that the point of symmetry had to lie at the intersection of these two asymptotes, namely the point (3; 6). Translating the graph with the transformation $y=f(x+3)-6$, I found that the transformed equation was point symmetric around (0; 0), implying that the original graph was indeed point symmetric at (3; 6) (see figure 10).

Using precisely the same reasoning as in the previous problem, I could also find a point of symmetry for the rational function $f(x)=(x^2+2x-24)/(x+2)$ at $(-2; -2)$, the intersection of its two asymptotes.

13. Further generalizations and proofs

As shown in the previous two problems I had made a number of intuitive conjectures which happened to work out in both cases. Was it by chance or could one prove that these were true in general?

However some false conjectures were also simultaneously made through over-generalization from special cases. At one stage from my experience with point symmetric rational functions like the above, I for instance thought that if a point symmetric function has a vertical or horizontal asymptote then it must also have at least one skew asymptote (or correspondingly a horizontal or vertical one). However upon trying to prove it in general, I soon realized that it was false by constructing a counter-example of a point symmetric function with only one vertical asymptote as shown by the first figure in Figure 11, as well as a point symmetric function with only

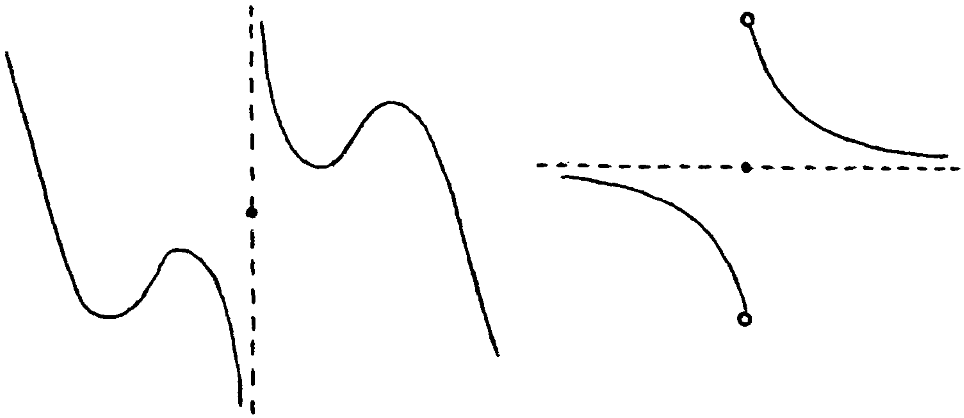

Figure 11.

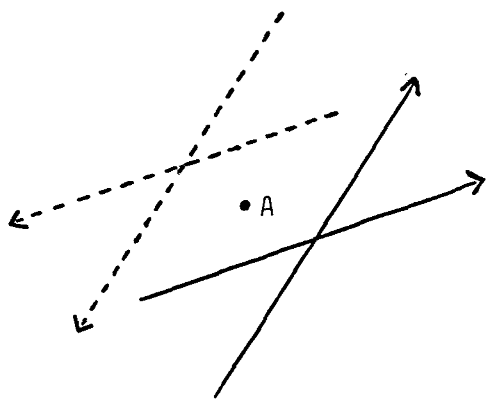

Figure 12.

a horizontal asymptote as shown by the second figure in Figure 11. Eventually the following general conjecture was made and proved.

Theorem 4. If a point symmetric function has 2 and only 2 non-parallel asymptotes then its point of symmetry lies at the intersection of the two asymptotes.

Proof. Consider any two non-parallel asymptotes as shown by the solid lines in Figure 12. If the point of symmetry is not at the intersection of the two asymptotes, say at A as shown, the point of intersection would not be mapped on to itself upon a half-turn, but on to the point of intersection of two additional asymptotes (shown by the dotted lines). But this is a contradiction since only two non-parallel asymptotes are given which can have only one point of intersection. Similarly, the choice of A on either of the asymptotes, but not at the intersection, leads to a contradiction since an additional asymptote and a point of intersection is added. Thus, the point of symmetry must lie at the intersection of the two asymptotes. (We can similarly prove that if a point symmetric graph has 1 and only 1 asymptote then its point of symmetry must lie on that asymptote.)

Theorem 5. A function of the type $y=f(x)/g(x)$ is reflective symmetric around $x=a$ if and only if: (1) both $f(x)$ and $g(x)$ are reflective symmetric around $x=a$, or (2) both $f(x)$ and $g(x)$ are point symmetric around $(a;0)$.

Proof. If $y=f(x)/g(x)$ is reflective symmetric around $x=a$ it can be translated to lie symmetric around the y-axis. Thus,

$$\frac{f(x+a)}{g(x+a)} \Leftrightarrow \frac{f(-x+a)}{g(-x+a)} \Leftrightarrow \frac{-f(-x+a)}{-g(-x+a)}$$

Therefore, if we assume that $f(x)$ and $g(x)$ have no common factors other than 1 or -1, we can deduce that $f(x+a)\Leftrightarrow f(-x+a)$ and $g(x+a)\Leftrightarrow g(-x+a)$ or $f(x+a)\Leftrightarrow -f(-x+a)$ and $g(x+a)\Leftrightarrow -g(-x+a)$.

Consequently $f(x)$ and $g(x)$ are both reflective symmetric around $x=a$ or $f(x)$ and $g(x)$ are both point symmetric around $(a;0)$. This then completes the proof of the forward implication. The converse can now be proved in the same way, but without the restriction that $f(x)$ and $g(x)$ have no common factors other than 1 or -1.

Using similar reasoning and the same proof technique, I next considered the analogous case for a point symmetric graph of the same form. This led to the following discovery.

Theorem 6. A function of the type $y=f(x)/g(x)$ is point symmetric around $(a;0)$ if and only if: (1) $f(x)$ is reflective symmetric around $x=a$ and $g(x)$ is point symmetric around $(a;0)$ or (2) $f(x)$ is point symmetric around $(a;0)$ and $g(x)$ is reflective symmetric around $x=a$.

Proof. Similar to that of Theorem 5 and is left to the reader.

14. Further applications

As special cases of Theorems 5 and 6 we can consider firstly the reciprocal functions of the type $y=1/g(x)$. Since $y=1$ is reflective symmetric around any $x=a$ but not point symmetric around any $(a;0)$, it follows from Theorem 5 that a reciprocal function is reflective symmetric around $x=a$ if and only if $g(x)$ is reflective symmetric around $x=a$. An example is given by the first figure in Figure 13 where $y=1/(x^2-2x+3)$ is reflective symmetric around $x=1$ since x^2-2x+3 is reflective symmetric around $x=1$. Similarly from Theorem 6 a reciprocal function is point symmetric around $(a;0)$ if and only if $g(x)$ is point symmetric around $(a;0)$. For instance, $y=\operatorname{cosec} x=1/\sin x$ is point symmetric around $(0;0)$ since $\sin x$ is point symmetric around the origin. (See the second figure in Figure 13.) Similarly we can conclude that $y=1/(x-1)^3$ is point symmetric around $(1;0)$ since $(x-1)^3$ is point symmetric around $(1;0)$, but $y=1/(x^3+1)$ is not point symmetric around the point $(-1;0)$ since x^3+1 is point symmetric around $(0;1)$. (Even though a rough sketch of $y=1/(x^3+1)$ as shown in Figure 14 might suggest that it has a point of symmetry there.)

These additional theorems are also useful in directly determining vertical lines or points of symmetry in functions of the type $y=f(x)/g(x)$ where $f(x)\neq 1$. For instance, $y=(x^3-3x^2+2x)/(x^3-3x^2+3x-1)$ is reflective symmetric around $x=1$, since the numerator and denominator are both point symmetric around $(1;0)$ (Theorem 5). Similarly according to the same theorem the function $y=(x^2-2)/(-x^2+2x)$ cannot be reflective symmetric around any vertical line since the numerator and denominator are not reflective symmetric around the same vertical line. Using

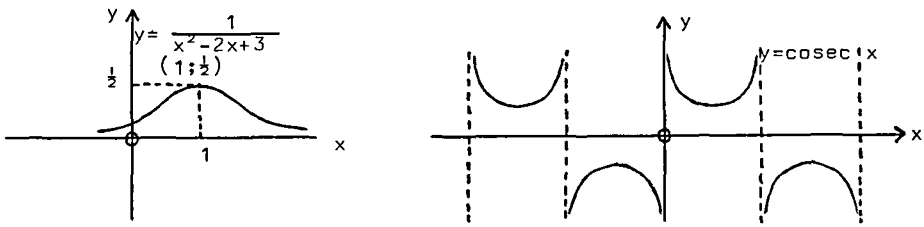

Figure 13.

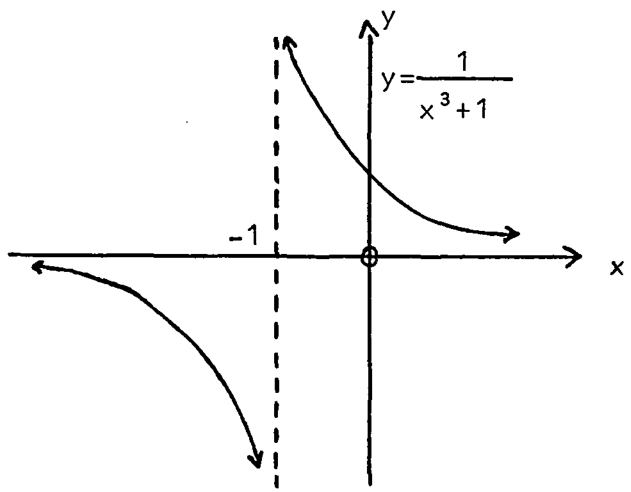

Figure 14.

Theorem 6 we can immediately deduce that $y=(2x^2+4x+3)/(x+1)$ is point symmetric around $(-1; 0)$.

Since Theorem 6 provides no criteria for determining points of symmetry at any point $(a; b)$ in comparison to the strong criteria for reflective symmetry around $x=a$ as contained in Theorem 5, it is generally somewhat more difficult to determine points of symmetry of graphs beforehand. For instance, let us consider the graph of the function $y=f(x)=(x^3-x^2-x-1)/(x^2-2x)$.

The numerator is point symmetric around $(1/3; -38/27)$ and the denominator is reflective symmetric around $x=1$, and the function can therefore not be reflective symmetric. However, if we determine its derivative

$$\frac{dy}{dx}=\frac{x^4-4x^3+3x^2+2x-2}{(x^2-2x)^2}$$

we find that the numerator and denominator are both reflective symmetric around $x=1$. Therefore, dy/dx is reflective symmetric around $x=1$ according to Theorem 5. But according to Theorem 2, $y=f(x)$ will then be point symmetric somewhere along the line $x=1$. Since the function is defined for this value, the point of symmetry will lie on the function itself (Theorem 3), namely at $(1; 2)$. The graph of this function is shown in Figure 15.

For another example, let's consider the function defined by $y=f(x)=x/(x-1)(x+2)$, the graph of which is shown in Figure 16. When I first saw this graph in

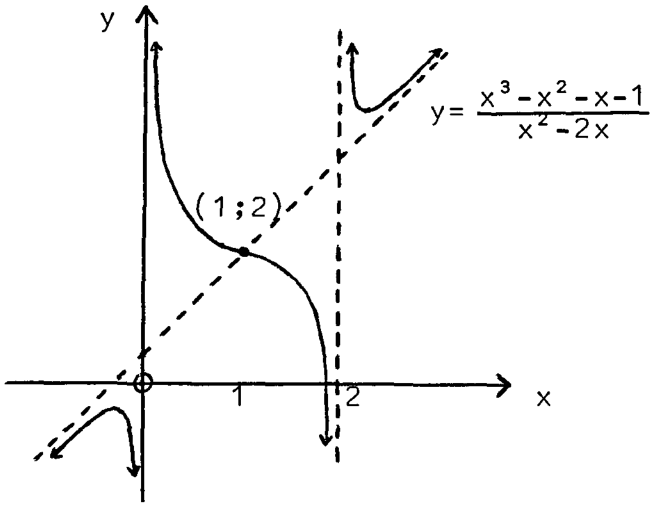

Figure 15.

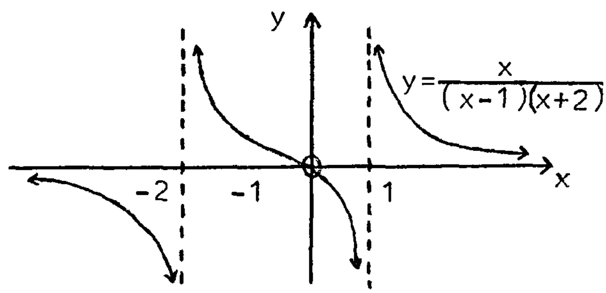

Figure 16.

Allendoerfer and Oakley ([2], p. 182), I intuitively felt that this graph was point symmetric somewhere between $x=0$ and $x=-1$, possibly at $(-1/2; 2/9)$. But one's intuition is not always right! Firstly, we can make no firm conclusion from the fact that the numerator is point symmetric around (0; 0) and the denominator is reflective symmetric around $x=-1/2$. (Note the similarities with the previous problem.) If we determine its derivative, we find

$$\frac{dy}{dx}=\frac{-x^2-2}{(x^2+x-2)^2}$$

Since the numerator and denominator are not reflective symmetric around the same vertical line, the derivative cannot be reflective symmetric around any $x=a$ (Theorem 5). Therefore, according to Theorem 2 we can conclude that this function has no point of symmetry. Actually in retrospect, there is a simpler way to prove that this function has no point of symmetry. For instance, since the function has 3 and only 3 asymptotes respectively at $x=-2$; $x=1$ and $y=0$, it cannot lie at $(-1/2; 2/9)$ since a half-turn of the function around this point would introduce another horizontal asymptote. To be symmetric with respect to the asymptotes it would therefore have to lie at $(-1/2; 0)$, but this leads to another contradiction since the function is defined at $x=-1/2$ and must therefore lie on the function according to Theorem 3.

One should also be careful in situations where common factors are involved and not make hasty conclusions. Consider for example the rational function $y=(x-1)^2(x^2+4x+28)/(x-1)^2(x+2)^2$. As the numerator and denominator are both reflective symmetric around $x=-1/2$, one might hastily conclude according to Theorem 5 that this function is reflective symmetric around $x=-1/2$. However, as the function is undefined for $x=1$ and $x=-2$, and since these values are unsymmetric with respect to $x=-1/2$, neither the numerator, denominator nor the function itself can therefore be reflective symmetric around $x=-1/2$. (Note however that $y=(x^2+4x+28)/(x+2)^2$ is reflective symmetric around $x=-2$.)

15. Still further results

As is usually the case with mathematics, new problems generate still further problems, and new theorems suggest still further theorems. It is a never-ending evolution, which is sometimes best likened to a nuclear chain reaction! During the investigation described above, I had for example initially only focused on proving the point symmetry of all cubic polynomials. However, in the process I recognized a relationship between the symmetric properties of the differentiable parts of a function and the symmetric properties of its derivative, which culminated in Theorems 1 and 2. Although their application to polynomial functions was straightforward, I was led to discover two more theorems when I tried to extend their application to more complex functions (e.g. rational). But if one could consider the symmetric properties of the quotients of functions, why not also the symmetric properties of the sums and products of functions? The results which follow were mainly discovered in a formal manner, arguing analogously from the proof techniques employed earlier. In contrast to the role that specific examples played in the discovery of the earlier results, they were now used mainly to check the conquests of deduction afterwards. Although these results are perhaps not quite as useful as the earlier ones since they are not biconditional, they nevertheless nicely retain the 'duality' between vertical line and point symmetry.

Theorem 7. If a function $f(x)$ is point symmetric around $(a;b)$ and a function $g(x)$ is point symmetric around $(a;c)$ then the function $y=f(x)\pm g(x)$ will be point symmetric around $(a;b\pm c)$.

Proof. If $f(x)$ is point symmetric at $(a;b)$ and $g(x)$ is point symmetric at $(a;c)$, then

$$f(x+a)-b=-f(-x+a)+b \qquad (6)$$

and

$$g(x+a)-c=-g(-x+a)+c \qquad (7)$$

Now consider the function $y=f(x)+g(x)$ and its translation by a horizontal units and $b+c$ vertical units. Therefore, the translated function is

$$f(x+a)+g(x+a)-(b+c)=f(x+a)-b+g(x+a)-c$$

$$=-f(-x+a)+b-g(-x+a)+c$$

by substituting equations (6) and (7)

$$=-[f(-x+a)+g(-x+a)]+(b+c)$$

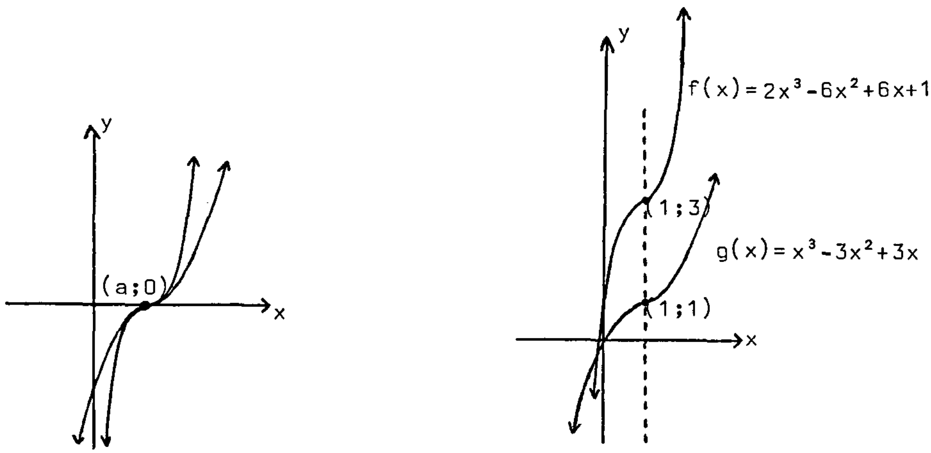

Figure 17.

which means that $y=f(x)+g(x)$ is point symmetric around $(a; b+c)$. Since we can in a similar fashion prove that $y=f(x)-g(x)$ will be point symmetric around $(a; b-c)$, this completes the proof of the theorem.

In the discovery of this particular result, I first considered the case where $f(x)$ and $g(x)$ are point symmetric around the same point $(a; 0)$ on the x-axis. (See for example the first figure in Figure 17.) However, upon proving in general for such cases that $f(x)\pm g(x)$ will be point symmetric around $(a; 0)$, I realized that it was not necessary that both be point symmetric around $(a; 0)$. (Compare with Theorem 3.) This then led to the general result formulated as Theorem 7 above (see second figure in Figure 17 where $f(x)+g(x)$ is point symmetric around $(1; 4)$). The converse of this theorem is of course false, since $f(x)$ and $g(x)$ need not both be point symmetric nor point symmetric on the same vertical axis $x=a$ for a function $f(x)\pm g(x)$ to be point symmetric around $(a; b)$. For example, $y=(2x^3)+(x^3-3x^2)=3x^3-3x^2$ is point symmetric around $(1/3;\ -2/9)$, but its component parts $2x^3$ and x^3-3x^2 are respectively point symmetric around $(0; 0)$ and $(1; -2)$.

16. Logical interdependence

Theorem 8. *If two functions $f(x)$ and $g(x)$ are both reflective symmetric around $x=a$, then the function $y=f(x)\pm g(x)$ will also be reflective symmetric around $x=a$.*

Proof. Although this result can be proved from first principles just like the previous theorems, the logical relationship between these theorems are nicely emphasized by the following proof.

Consider firstly the functions $f(x)$ and $g(x)$ which are both reflective symmetric around $x=a$. Then according to Theorem 1 $f'(x)$ and $g'(x)$ will both be point symmetric around $(a; 0)$. Therefore according to Theorem 7, $f'(x)\pm g'(x)$ will be point symmetric around $(a; 0)$, but since this is equivalent to the derivative of $y=f(x)\pm g(x)$, according to Theorem 1, the function $y=f(x)\pm g(x)$ must be reflective symmetric around $x=a$.

This result is clearly analogous to Theorem 7, and as shown in Figure 18, one immediately anticipates that $f(x)\pm g(x)$ would still be reflective symmetric around $x=-1$. Note that in both theorems, if we consider the special case where either $f(x)$

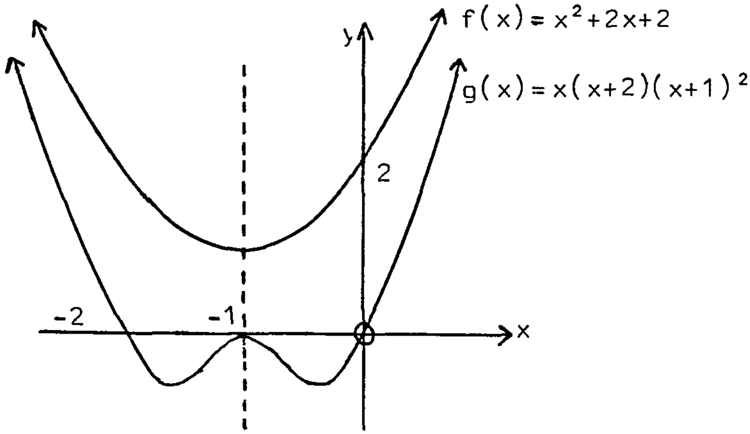

Figure 18.

or $g(x)$ is not a function of x but only a constant, say k, we have the well known result that the respective graphs are translated by k units in the vertical direction without any change to their symmetric properties. Furthermore, it should be clear that Theorem 8 provides a sufficient condition for vertical line symmetry, but is not a necessary condition, thereby implying that the converse is false. For example, the addition of two quadratic functions with different vertical lines of symmetry can easily provide another quadratic function with vertical line symmetry, not necessarily coinciding with either line of symmetry of its original components. Note however that in general it is not a sufficient condition for a function $f(x) \pm g(x)$ to be reflective (or point) symmetric, when $f(x)$ and $g(x)$ are reflective symmetric (or point) symmetric around (or on) different vertical lines. For a simple example, consider the function $y = f(x) - g(x)$ with $f(x) = x^2 + 2x$ and $g(x) = x^2$. This function is clearly not reflective symmetric around a vertical line, even though $f(x)$ and $g(x)$ are. Similar counter-examples can easily be produced for the general addition/subtraction of point symmetric functions.

Theorem 9. If a function $f(x)$ is reflective symmetric around $x = a$ and a function $g(x)$ is point symmetric around $(a; 0)$ or if $f(x)$ is point symmetric at $(a; 0)$ and $g(x)$ is reflective symmetric around $x = a$, then the function $y = f(x) \cdot g(x)$ is point symmetric at $a; 0)$.

Proof. Can be done directly from first principles as in the previous examples and is left to the reader.

Theorem 10. If functions $f(x)$ and $g(x)$ are both reflective symmetric around $x = a$ or both are point symmetric around $(a; 0)$, then the function $y = f(x) \cdot g(x)$ is reflective symmetric around $x = a$.

Proof. Although this result can also be proved from first principles, it is useful for the purpose of systematization to prove it in terms of the other theorems as follows.

Consider functions $f(x)$ and $g(x)$ which are both reflective symmetric around $x = a$, as well as the function $y = f(x) \cdot g(x)$ and its derivative $dy/dx = f'(x) \cdot g(x) + f(x) \cdot g'(x)$. Since $f(x)$ is reflective symmetric around $x = a$, $f'(x)$ will be point symmetric around $(a; 0)$ (Theorem 1). Therefore, according to Theorem 9

$f'(x)\cdot g(x)$ will be point symmetric around $(a;0)$. Similarly we can show that $f(x)\cdot g'(x)$ is also point symmetric around $(a;0)$, and employing Theorem 7 we can deduce that $dy/dx = f'(x)\cdot g(x) + f(x)\cdot g'(x)$ is point symmetric around $(a;0)$. However, according to Theorem 1, if dy/dx is point symmetric around $(a;0)$ we can deduce that $y=f(x)\cdot g(x)$ is reflective symmetric around $x=a$. In a similar fashion we can also prove the case where $f(x)$ and $g(x)$ are both point symmetric around $(a;0)$. This then completes the proof of this theorem.

The converses of Theorems 9 and 10 are both false, since the theorems do not provide necessary conditions for vertical line or point symmetry. For instance, consider $y=(x-1)(x-3)=x^2-4x+3$ where the factors are respectively point symmetric around $(1;0)$ and $(3;0)$ (among others) but the product x^2-4x+3 is reflective symmetric around $x=2$, thus providing a counter-example to the converse of Theorem 10. (Note however that in general it is not a sufficient condition for a function $f(x)\cdot g(x)$ to be reflective symmetric when $f(x)$ and $g(x)$ are point symmetric around different points on the x-axis. Consider for example $y=x^3(x-1)=x^4-x^3$ which is not reflective symmetric although its factors are respectively point symmetric around $(0;0)$ and $(1;0)$.) As direct applications of Theorems 9 and 10 we may consider the first two figures already given in Figure 10. These theorems may also be employed in combination with some of the other theorems to prove, for example, the converse of Theorem 5 in terms of Theorem 6. (It is not possible, for instance, to prove the forward implication of Theorem 5 using Theorems 7 to 10 (among others) since their converses are false.)

Finally, the following intuitive conjectures are left for the reader to investigate (i.e. to prove, improve or refute):

> If a periodic function has an unrestricted domain and a point of symmetry and/or a vertical line of symmetry, then it has infinitely many points of symmetry and/or vertical lines of symmetry.
>
> If a point symmetric function is smooth and continuous at its point of symmetry, then its point of symmetry is a point of inflection.
>
> If a function with vertical line symmetry is smooth and continuous at the point of intersection between its line of symmetry and the function, then this point is either a local maximum or minimum or the function is constant at this point.

17. Final remarks

Although I have attempted to write as faithfully about the way I arrived at the results reported in this paper, it should be noted that it is not a hundred per cent accurate. Besides cutting corners for the sake of economy here and there, one has the problem that writing is strictly sequential and uni-dimensional while one's thinking is non-sequential and multi-dimensional: one can think of many things and their interrelationships at the same time, but one cannot write them down in that way.

Therefore I hope that I have allowed the reader at least partially to share in the excitement and personal joy I experienced during this investigation. Hopefully this paper has also convincingly demonstrated the role of 'quasi-empirical' methods in mathematical research. It has shown that the discovery and invention of new mathematics does not start with the fully fledged theorem, but frequently grows from the need to solve a particular problem situation and through several generalizations and refutations. Formal deductive reasoning was mostly preceded by intuitive leaps and bounds deeply embedded in a context of conjecturing and quasi-empirical checking. The educated guess, the intuitive leap furthermore, does not come from nowhere but comes from varied experiences and a quasi-empirical familiarization with a specific topic over a period of time. Thus we saw that confidence in the truth of conjectures 1 and 2 gradually grew from the consideration of several specific examples and their visual representation to the eventual formulation of an intuitive geometric proof. This was followed by a formal analytical proof whose main purpose was not the establishment of their truth, but for the sake of systematizing them into a well established body of knowledge.

Further quasi-empirical checking then revealed a subconscious restriction of their domain of validity, which led to their more general reformulation. In the consequent application of Theorems 1 and 2 to the finding of vertical lines or points of symmetry in specifically the rational functions, new conjectures were discovered and proved as Theorems 5 and 6. Using the proof techniques developed so far, this led to a mainly deductive investigation of the symmetric properties of the sums and products of functions, with specific examples now fulfilling a checking role, leading to Theorems 7 and 10. Note that Theorems 3 and 7 also nicely illustrated the discovery function of proof in another context, i.e. how new mathematical results are sometimes discovered through deductive generalization. For example, by identifying the essential characteristic of a conjecture by the production of a proof, one sometimes finds that the conditions contained in the original conjecture are sufficient, but not necessary, thus leading directly to a generalization. (Compare with [5].) Although at the time of writing I am not sure of the originality of any of the ideas expressed here, I nevertheless experienced it as an exciting personal learning experience and gained a better understanding of the nature of mathematics through it.

Acknowledgment

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