Waldemar Pompe <waldemar.pompe@gmail.com>

Problem about quadrilateral from Johan

Let $ABCD$ be a cyclic quadrilateral, where the diagonals intersect in E. Let I_1 and O_1 be the incentre and circumcentre of triangle ABE. Let I_2 and O_2 be the incentre and circumcentre of triangle CDE. Prove that

$$((AO_2)^2 - (BO_2)^2) - ((AI_2)^2 - (BI_2)^2) = ((DO_1)^2 - (CO_1)^2) - ((DI_1)^2 - (CI_1)^2).$$

Solution

Let R_2 be the circumradius of traingle CDE. Let $p(X; O, r)$ denote the power of the point X with respect to the circle with center O and radius r. Then we have:

$$(AO_2)^2 = p(A; O_2, R_2) + R_2^2 = AE \cdot AC + R_2^2\,,$$

$$(BO_2)^2 = p(B; O_2, R_2) + R_2^2 = BE \cdot BD + R_2^2\,.$$

Therefore

$$(AO_2)^2 - (BO_2)^2 = AE \cdot AC - BE \cdot BD$$

and similarly

$$(DO_1)^2 - (CO_1)^2 = DE \cdot BD - CE \cdot CA\,.$$

Let K, L, be the points where the incircle of CDE touches AC and BD, respectively. Similarly, let M, N, be the points where the incircle of ABE touches AC and BD, respectively. Then by Pythagoras

$$(AI_2)^2 - (BI_2)^2 = AK^2 - BL^2\,,$$

$$(DI_1)^2 - (CI_1)^2 = DN^2 - CM^2\,.$$

Now denote: $AB = a$, $BE = b$ and $AE = c$. Let t be the scale of similarity between triangles ABE and CDE. Then we have: $CD = ta$, $CE = tb$ and $DE = tc$. If $s = \frac{1}{2}(a + b + c)$, then the desired equality is equivalent to:

$$\Big(c(c + tb) - b(b + tc)\Big) - \Big((c + t(s - a))^2 - (b + t(s - a))^2\Big) =$$

$$= \Big(tc(tc + b) - tb(tb + c)\Big) - \Big((tc + (s - a))^2 - (tb + (s - a))^2\Big)\,.$$

And everything cancels out...

Remark. We didn't essentially use in the computation that $KN = s - a$. Therefore the problem can be generalized to the case, where I_1, I_2 lie on the angle bisector of $\angle AEB$ such that $AEBI_1$ is similar to $DECI_2$.