Generalizing a theorem of Arsalan Wares

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In mathematics it is often the case that a new result is obtained by looking at an old result in a new way and extending it to obtain a generalization of the old result. Hence it frequently happens that: Today's theorem becomes tomorrow's corollary.

Gary Chartrand, Albert Polimeni & Ping Zhang (2003: 107)

Introduction

New developments in mathematics and science are sometimes stimulated by revisiting and 'looking back' in the style of Polya (1945) at previously established results. In this way, Einstein made an important breakthrough in physics by re-examining the Michelson-Morley experiment, and coming to the conclusion that the ether didn't exist and that light always travelled at a constant speed, irrespective of the speed or position of the observer.

Similarly, continued attempts to find general algebraic solutions for the quintic equation finally led to the development of abstract algebra, and entirely new concepts such as groups and fields. The re-examination of the axioms of Euclid's *Elements*, identified several unstated assumptions, eventually resulting in Hilbert's expansion of the Euclidean axioms to fifteen.

On a smaller scale in the classroom, it is valuable to identify suitable problems for students that can be used to encourage further reflection that leads to variations or generalizations of results. Such experiences may help illustrate the important 'mathematical habit of the mind' of 'generalization', which Johnston-Wilder & Mason (2005, p. 93) have called as lying at the very 'heart of mathematics'.

Wares' Theorem

Arsalan Wares (2010) gives the following interesting result he discovered in a dynamic geometry environment: Choose arbitrary points X, Y and Z, respectively on sides BC, AC and AB of ΔABC. If perpendiculars are dropped from A, B and C to line segments YZ, XZ and XY, respectively, and if U, V and W are the intersections of these perpendiculars as indicated in Figure 1, then ΔUVW is similar to ΔXYZ.

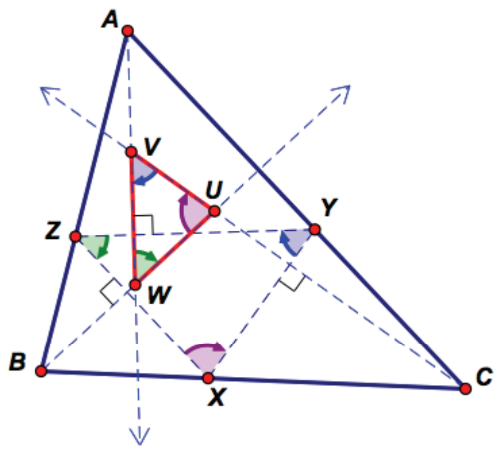

Figure 1

Proof
The proof follows quite easily by calculating angles. For example, $\angle WVU = \angle VAC + \angle VCA$ (exterior angle of Δ).

But $\angle ZYA = 90° - \angle VAC$
and $\angle XYC = 90° - \angle VCA$.

Hence, since AYC is a straight line, $\angle ZYX = \angle VAC + \angle VCA$ which implies that $\angle WVU = \angle ZXY$. In the same way can be shown that the other corresponding angles are equal, and hence that the triangles are similar.

What if?
An effective way of getting students to start posing their own questions is to regularly encourage them to ask

'what-if?' questions when encountering new theorems or problems, and then to get them to explore these further with dynamic geometry. For example, what happens if X, Y and Z are chosen on the extensions of the sides? Is the result still valid? Perhaps even bolder, one could ask what happens if X, Y and Z are not on the sides of the triangle? Does the result still hold? If true in any of the aforementioned, does the same proof still hold or does it need modification?

In *Sketchpad* it is easy to explore the last question straight away as it has a nice, powerful feature for this sort of thing without the need to make a completely, new construction as might be the case with other software. One need only select say point X and then the line segment AB on which it lies, and then simply choose the "*Split Point from Segment*" in the **Edit** menu. Much to my surprise, I found that the result remained true despite 'splitting' the points from the sides of the triangle. So the condition that the points X, Y and Z were on the sides of the triangle were not necessary at all! So the theorem is really about two arbitrary triangles, ABC and XYZ, with perpendiculars dropped from the vertices of ABC to the sides of XYZ.

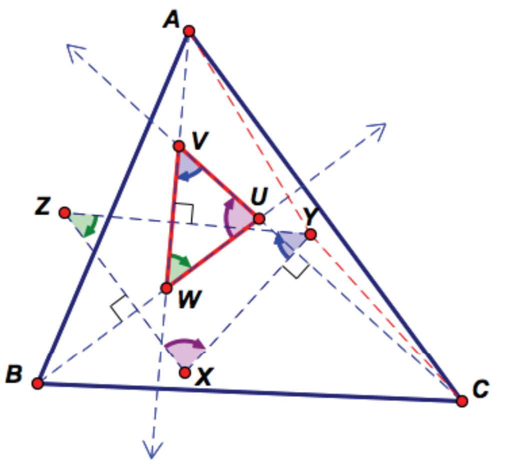

Figure 2

Proof
The previous proof only needs some slight modification, and a little more 'angle-chasing', the only difference being that AYC is no longer a straight line. For example, in Figure 2, $\angle WVU = \angle VAC + \angle VCA$ (exterior angle of Δ) still as before.

But, now $\angle ZYA = 90° - \angle VAY$ and $\angle XYC = 90° - \angle VCY$.
Since $\angle YCA = \angle VCA - \angle VCY$
and $\angle YAC = \angle VAC - \angle VAY$,

it follows in ΔAYC that $\angle AYC$ has size:

$180° - (\angle VCA - \angle VCY + \angle VAC - \angle VAY)$.

Adding up and subtracting angles ZYA, XYC and AYC surrounding point Y from 360º, give us as before $\angle ZYX = \angle VAC + \angle VCA$, which implies $\angle WVU = \angle ZXY$. In the same way, it can be shown that the other corresponding angles are equal, and hence that the triangles are similar.

Generalizing Perpendiculars to Equi-inclined Lines

One way of generalizing perpendicular lines as shown in De Villiers (2002) to generalize the theorems of Neuberg and Viviani, as well as the Simson line, is to consider constructing lines so that they make *equal angles* with the sides. What if we construct such equi-inclined lines here? Is the result still valid?

This can also easily be done with dynamic geometry by rotating each of the perpendiculars by the same variable (directed) angle, and considering the intersections of the rotated lines. A quick investigation confirmed that the result still holds. In order to prove the result, the following Lemma is useful (and perhaps interesting in its own right).

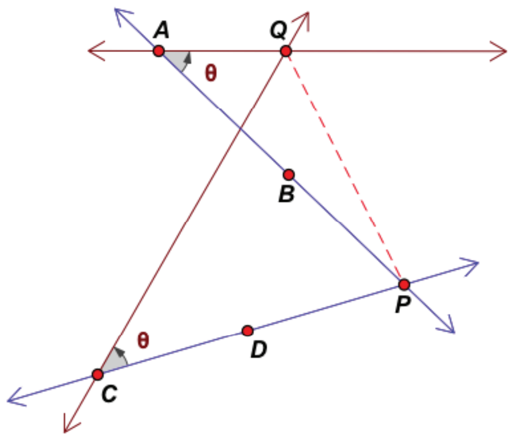

Figure 3

Lemma
If two lines AB and CD meet in P, and they are respectively rotated by the same directed angle around centres A and C, then the rotated lines meeting at Q, are inclined at the same angles as they were at P (see Figure 3).

Proof of Lemma
Since it is given that the rotation angles BAQ and DCQ are equal, and they are subtended by the same segment PQ, it follows that $ACPQ$ is cyclic. Hence, angle AQC = angle CPA on chord AC, and therefore their supplements are equal as well.

Proof of Equi-inclined Lines
Using the above lemma, it is now straightforward to prove the general result (see Figure 4). The perpendiculars from A and B, intersecting at W, after being respectively rotated around A and B by the same angle, now intersect at M. Hence, according to the

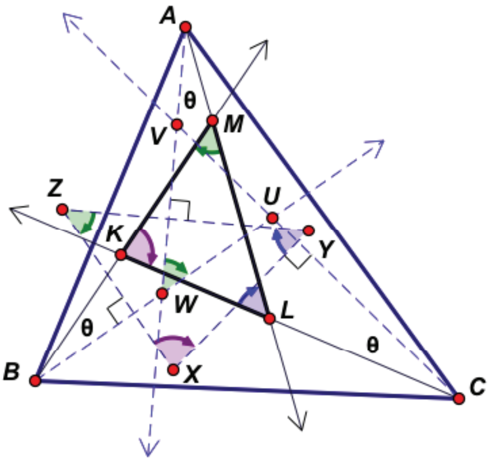

Figure 4

lemma, the two pairs of angles at *M* are equal to those around *W*. In the same way, the other angles at *K* and *L* are respectively equal to *U* and *V*. Thus, ΔKLM is similar to ΔUVW and therefore to ΔXYZ also.

Further generalization to polygons

Since the result does not rely on any special properties of triangles, and only on angles, it is easy to see that the result further generalizes to quadrilaterals and higher polygons as shown in Figure 5 for perpendiculars drawn from *ABCD* to the sides of *EFGH*. In this case, however, *PQRS* is not similar to *EFGH*, but only has equal corresponding angles (since equality of angles is a sufficient condition for similarity, only for triangles).

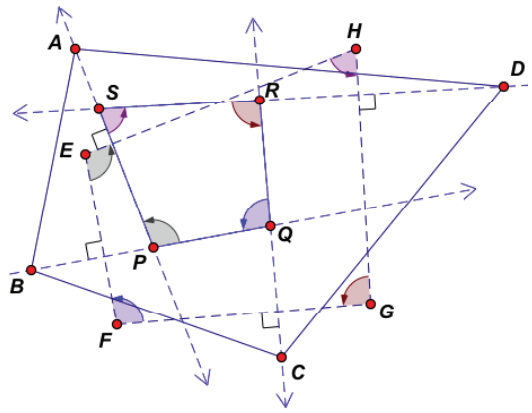

Figure 5

It might be a good exercise to ask students to again do some 'angle-chasing' to show one or more corresponding angles between *EFGH* and *PQRS* equal. Moreover, by using the same lemma as before, the result generalizes further if equi-inclined lines are drawn to the sides of *EFGH*.

Concluding comment

Doing an investigation such as this with students at high school or undergraduate level might help to encourage them not to immediately move on from one problem to the next, but perhaps to pause for a while in reflection on a solution. One may never know what further delights perhaps lie hidden in a result, however mundane or trivial it may appear, unless one has an open mind to ask some 'what-if' questions, and dare to explore variations and further generalizations. The general availability of dynamic geometry software makes such explorations exciting, and often fruitful as shown in this case.

References

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An interactive, dynamic Java sketch illustrating the results discussed here is available online at: http://dynamicmathematicslearning.com/wares-further.html

With the wisdom of hindsight the results in Figures 1, 2 and 5 can more easily and elegantly be proved using the 'exterior angle' of a cyclic quadrilateral theorem.

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