Equi-distant Theorems

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Theorem 1: If all the perpendicular bisectors of the sides of a polygon are concurrent, then it is cyclic.

Proof: First note that all points on a perpendicular bisector are equi-distant from the vertices of the side it perpendicularly bisects. Therefore, if all the perpendicular bisectors of a polygon are concurrent, it means there exists a unique point equi-distant from all the vertices. Thus, a circle can be drawn with the point of concurrency as center and radius equal to the distance to any one of the vertices, which implies it is cyclic.

Theorem 2 (converse of Theorem 1): If a polygon is cyclic, then all the perpendicular bisectors of its sides are concurrent.

Proof: If a polygon is cyclic, it means that there must exist a point equi-distant from all the vertices, namely, the center of the circle. This means that the center must lie on the perpendicular bisectors of all the sides (see note 1 in first proof), and shows all perpendicular bisectors pass through the center, i.e. are concurrent.

Theorem 3 (further generalization of Theorem 1): If $n - 1$ perpendicular bisectors of the sides of a n-gon are concurrent, then it is cyclic ($n > 2$).

Proof: Number the vertices of the polygon as $A_1, A_2, A_3, \ldots A_{n-1}A_n$. If we assume $n - 1$ perpendicular bisectors of the sides $A_1A_2, A_2A_3, \ldots A_{n-1}A_n$ are concurrent at P, then it is required to prove that the perpendicular bisector of A_nA_1 also passes through P. But from this concurrency, it follows that P is equi-distant from all the vertices $A_1, A_2, A_3, \ldots A_n$ by the transitive property. Therefore, P is also equi-distant from vertices A_1 and A_n and the perpendicular bisector of A_nA_1 will pass through P.

Note: The converse of Theorem 3 is trivially true. If a n-gon is cyclic, it has according to Theorem 2, all n of its perpendicular bisectors concurrent. Therefore, it has n-1 perpendicular bisectors concurrent.