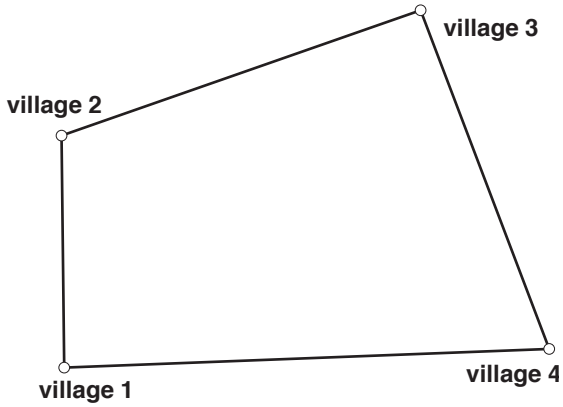
Water Supply I: Four Towns

In a developing country such as South Africa, there are many remote rural areas where people do not have access to safe, clean water and are dependent on nearby rivers and streams for their water supply. Apart from being unreliable due to frequent droughts, these rivers and streams are often muddy and unfit for human consumption. Suppose the government wants to build a water reservoir and purification plant for four villages in such a remote rural area. Where should the government place the water reservoir so that it is the same distance from all four villages?

INVESTIGATE



1. Before you work in Sketchpad, draw a point in the drawing that shows your best guess for the location of the reservoir. Label the point *P*.

- Open the sketch **Water Supply I.gsp,** which shows the map in the drawing.
- Construct a point *P* anywhere to represent the water reservoir.
- Measure the distances from the point *P* to each of the four vertices.
- Drag point *P* and observe the four distance measurements. Try to locate point *P* so that it is the same distance from all four vertices.

To measure the distance between two points, select both points and choose **Distance** from the Measure menu.

2. Were you able to locate point *P* so that the four distances were equal? If so, how does the location you found in the Sketchpad sketch compare with your guess in Question 1? If not, what did you discover in trying?

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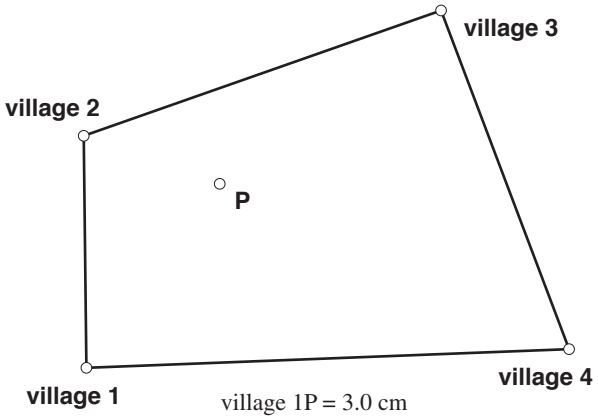
A Simpler Problem

How can you locate point *P* precisely without using trial and error and dragging? In problem solving, it is often useful to look at a *simpler case* of a problem. In the original problem, we wanted to find a point that is *equidistant* (at equal distances) from four vertices. A simpler case would be to look for a point (or points) equidistant from just two vertices.



Continue in the same sketch, but focus on only two adjacent vertices for now.

It may help to drag the two extra measurements off to the side or hide them, leaving behind only the measurements related to the two vertices you have chosen to focus on.

- If necessary, drag point *P* so that it is equidistant from two adjacent vertices.
- Select point *P* and choose **Trace Point** from the Display menu.
- Drag point *P* slowly for a few centimeters, keeping it as close to equidistant as possible from the two adjacent vertices.

3. Describe as many properties as you can that relate the traced path to the segment connecting the two vertices.

CHALLENGE Use your observations from Question 3 to revisit the original problem with all four villages. Come up with an alternative way to search for a point equidistant from all four vertices. Describe your construction method here. If you get stuck, continue reading.

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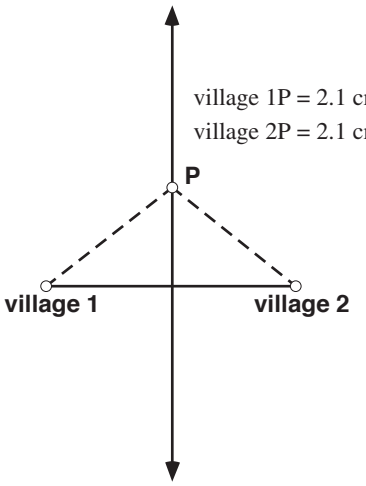
On the previous page, you should have found that there are an infinite number of points that lie equidistant from two vertices and that they all lie on a straight line. Furthermore, from symmetry you may notice that folding around this line of equidistant points maps one village onto the other; therefore, this line *bisects* the line segment connecting the villages and is perpendicular to it. This line of equidistant points is called the *perpendicular bisector* of a line segment.



To construct a perpendicular bisector in Sketchpad, first select the segment and choose **Midpoint** from the Construct menu. Then select the midpoint and the segment and choose **Perpendicular Line** from the Construct menu.

- If your villages have moved during your construction, use the action button to snap them to their correct locations.
- Construct all four perpendicular bisectors of the sides of the quadrilateral.

4. What do you notice about the four perpendicular bisectors of this quadrilateral?

A More General Problem

5. Do you think you can always find a point equidistant from all four vertices of a quadrilateral, no matter what the shape or size of the quadrilateral? Explain.

6. Drag any vertex of the quadrilateral. What do you notice about the perpendicular bisectors?

7. Do you still agree with your answer to Question 5? Explain.

- Use the button in your sketch to return the villages to their correct locations.
- Drag point *P* to the point that is equidistant from all four vertices.
- Construct a circle with *P* as its center and a village on the circumference.

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8. Record what you observe about the other vertices of the quadrilateral and explain why this must be true.

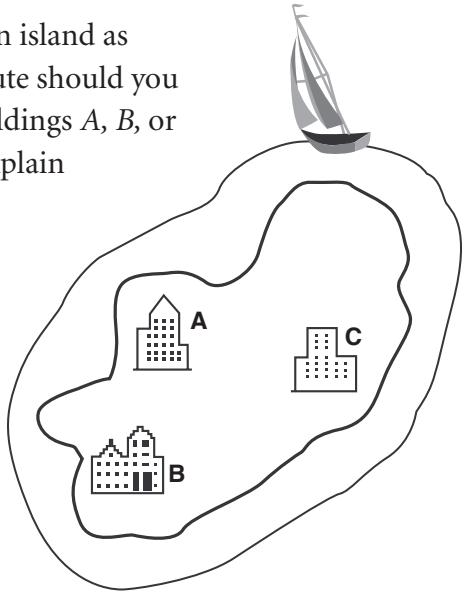
Further Exploration

1. In a new sketch, construct a quadrilateral and a central point so that the point is always equidistant from all four vertices. Make the quadrilateral as general as possible. Make sure the central point is always equidistant from the vertices, no matter which points you drag. Explain your construction method.

2. The dynamic Sketchpad scale drawing of the four villages is an example of a mathematical model that can be used to represent and analyze real-world situations. However, real-world situations are extremely complex and usually have to be simplified before mathematics can be meaningfully applied to them. What are some of the assumptions that could have been made above to simplify the original problem?

3. Suppose there is no equidistant point for four villages (that is, the perpendicular bisectors are not concurrent). Investigate what might be the "best" position to now place the water reservoir. Can you mathematically explain why you think that would be the "best" position?

4. Suppose you are sailing on a boat around an island as indicated here. At what locations on the route should you be to be able to determine which of the buildings *A*, *B*, or *C* is the tallest and which is the shortest? Explain your reasoning. (All buildings stand on the same level and are visible anywhere along the indicated route.)



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b. The sum $h_1 + h_2$ is constant, since the distance between the two opposite parallel sides is constant. Similarly, $h_3 + h_4$ is constant. Therefore, $h_1 + h_2 + h_3 + h_4$ is constant (equal to the sum of the two distances between the pairs of opposite sides). The result is generalizable to any polygon with an even number of sides and opposite sides parallel but not necessarily equal (that is, a *parallel-2n*-gon ($n > 1$)), where the sum of the $2n$ distances to the sides will be equal to the sum of the n distances between the pairs of opposite sides.

Another possibility to consider is the generalization to three dimensions (and more). Since the 3D analog of a triangle is a tetrahedron, students may first want to consider a regular tetrahedron. Instead of working with areas and distances, they will now need to work with volumes and areas. After further reflection, they should realize that the sum of the distances to the four faces of a tetrahedron, with all four faces having the same area a, would also be constant. For example, a point P inside the tetrahedron divides it into four tetrahedra, so that

$$\frac{1}{3}aH = \frac{1}{3}a(h_1 + h_2 + h_3 + h_4) \longleftrightarrow H = h_1 + h_2 + h_3 + h_4$$

Note that if the area of each face is the same, the height H from each face to the opposite vertex must also be the same, since its volume is constant. But this does not imply that the tetrahedron is necessarily regular.

4. The sum of the distances to the sides of an equi-angled pentagon is also constant. The result can be generalized to any equi-angled polygon (see Sharygin 2000, 50).
5. Some of the main assumptions are:

- The three beaches are perfectly straight.
- The island is perfectly equilateral.
- There are no other natural obstacles such as rivers, swamps, or dangerous animals that the surfer may want to avoid.
- The space between the beaches is flat everywhere (that is, there are no hills or valleys).
- The three beaches are equally good for surfing (which is why she visits each with equal frequency).

WATER SUPPLY I: FOUR TOWNS (PAGE 27)

The two activities Water Supply I: Four Towns and Water Supply II: Three Towns are real-world problems structured to introduce students to these concepts: the perpendicular bisector, the circumcenter, the theorem involving the concurrency of the perpendicular bisectors of any triangle, and the explanation (proof) of that theorem. Although the activities could be done independently of one another, it is not recommended. Students are given a quadrilateral problem first so they can discover that the perpendicular bisectors of a polygon are not necessarily always concurrent, and we hope they will experience some surprise in the second Water Supply activity when they find that for a triangle the perpendicular bisectors are always concurrent.

Prerequisites: None.

Sketches: Water Supply I.gsp and **Water Supply II.gsp.** Additional sketch is **Water Supply Ib.gsp.**

INVESTIGATE

Answers to Questions 1 and 2 will vary. Students are expected to first measure the distances from point P to the four vertices and then to drag point P until all four distances are equal. You might point out to your students that this is essentially the *guess-and-check* problem-solving strategy.

A SIMPLER PROBLEM

This part of the activity is intended to develop the idea that the perpendicular bisector is the set of points equidistant from the endpoints of a line segment.

3. Students should notice the following:
   a. Any point on the traced path is equidistant from the two vertices.
   b. The traced path is perpendicular to the line segment connecting the two vertices.
   c. The traced path bisects the line segment connecting the two vertices.
4. When the vertices are in their original places, the four perpendicular bisectors of the sides of the quadrilateral are concurrent.

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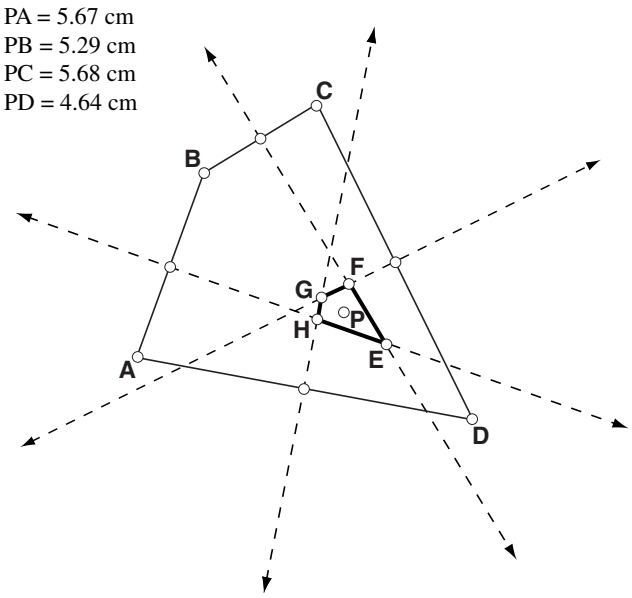
A More General Problem

5. Answers will vary. At this stage, students may believe that the perpendicular bisectors of any quadrilateral are concurrent. In Question 6, the students will discover that they're not.
6. The perpendicular bisectors of the sides of a quadrilateral are not necessarily concurrent.
7. Answers will vary.
8. Since all the towns are equidistant from point *P,* they must lie on a circle centered at *P.*

Further Exploration

1. Construct a circle and any four points on its circumference. These four points are all equidistant from the center, since the circle has constant radius.
2. Some of the main assumptions are
   - The four villages all lie on the same plane.
   - The space between the villages is flat everywhere (that is, there are no hills or valleys).
   - There are no other natural obstacles such as rivers, swamps, or dangerous animals that people may want to avoid.
   - The villages are all the same size (because if they were not, it might be much better to place the reservoir closer to the largest village).
3. First, it seems reasonable to conjecture that the "best" point ought to be located somewhere in the feasible region formed by the four respective circumcenters *E, F, G,* and *H* of the four triangles *ABC, BCD, CDA,* and *BAD.* Second, since we want to attempt to minimize the differences between all the distances, we need to minimize the sum of the absolute values of the differences to obtain the "best" point, for example,

$$|PA - PB| + |PA - PC| + |PA - PD| + |PB - PC| + |PB - PD| + |PC - PD|$$



$|PA - PB| + |PA - PC| + |PA - PD| + |PB - PC| + |PB - PD| + |PC - PD| = 2.86$ cm

An example sketch is given in **Water Supply Ib.gsp** where the "best" position can be obtained by dragging *P* until the desired minimum is obtained.

4. To solve this problem, students first have to realize that buildings farther away appear smaller. Therefore to make a visual comparison between the respective heights of any two buildings, they need to be equidistant from both buildings. The required positions can therefore be obtained by constructing the perpendicular bisectors of the sides *AB, BC,* and *CD* and finding the respective intersections with the route.

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