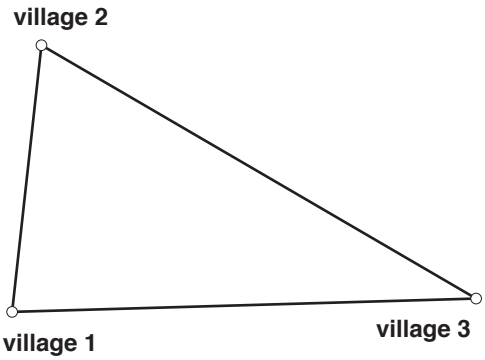
Water Supply II: Three Towns

The government needs to build a water reservoir for the three villages shown below. Where should the water reservoir be placed so that it is equidistant from all three villages?

INVESTIGATE



1. Before you work in Sketchpad, draw a point in the drawing that shows your best guess for the location of the reservoir. Label the point *P*.

- Open the sketch **Water Supply II.gsp,** which shows the map in the drawing.
- Construct the perpendicular bisectors of two sides of the triangle to locate the reservoir correctly in your sketch.

To construct a perpendicular bisector in Sketchpad, first select the segment and choose **Midpoint** from the Construct menu. Then select the midpoint and the segment and choose **Perpendicular Line** from the Construct menu.

2. How does your reservoir location compare to the location in your initial guess?

A More General Problem

3. Do you think you can always find a point equidistant from all three vertices, no matter what the shape or size of the triangle? Explain.

4. Drag any vertex of the triangle. What do you notice about the perpendicular bisectors? Do you still agree with your answer in Question 3? Explain.

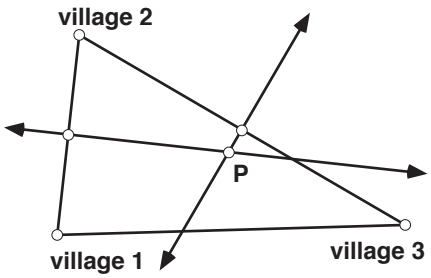
© 2012 Key Curriculum Press

EXPLAINING

In the activity Water Supply I: Four Towns, you found that the perpendicular bisectors of a quadrilateral do not always meet in one point; in other words, the perpendicular bisectors of the sides of a quadrilateral are not always concurrent. However, in that activity you should have discovered the rather surprising result that the perpendicular bisectors of any triangle are always concurrent at a point equidistant from all three vertices. This point of concurrency is called the *circumcenter* of the triangle since it is the center of the circle (the *circumcircle*) that passes through all three vertices.

If you came up with your own explanation as to why the perpendicular bisectors of any triangle are concurrent, compare it with the one that follows. If not, work through this investigation:

Let *P* be the point of intersection of two of your perpendicular bisectors. We will show logically that this point *P* must also lie on the perpendicular bisector of the third side; that is, all three perpendicular bisectors of a triangle always meet in the same point.



5. Pick one of the two perpendicular bisectors. What can you say about all the points on this bisector?

6. What can you say about all the points on the other perpendicular bisector?

7. What can you therefore say about *P,* the point of intersection of both perpendicular bisectors?

8. What can you therefore conclude about *P* and the perpendicular bisector of the third side?

© 2012 Key Curriculum Press

Present Your Explanation

Summarize your explanation using your answers to Questions 5–8. Your summary may be in paper form or electronic form and may include a presentation sketch in Sketchpad. You may want to discuss the summary with your partner or group.

Further Exploration

1. When is the circumcenter of a triangle inside, outside, or on the perimeter of a triangle?

2. Construct a general quadrilateral *ABCD* and any three of its perpendicular bisectors. Construct the intersection of two of these perpendicular bisectors and use it as a center to construct a circle that always passes through three of the vertices.

a. Drag the vertices of the quadrilateral until all three perpendicular bisectors are concurrent. What do you notice?

b. Drag the quadrilateral to a different shape until all three perpendicular bisectors are again concurrent. Also, construct the fourth perpendicular bisector. What do you notice?

c. In the space below, write a conjecture about your observations. Can you explain why it is true? Can you generalize further to pentagons, hexagons, and so on? Discuss with your partner or group.

© 2012 Key Curriculum Press

WATER SUPPLY II: THREE TOWNS (PAGE 31)

In this activity, students investigate a real-world problem and discover that the perpendicular bisectors of the sides of a triangle are concurrent. Although this activity could be done independently, it is intended to follow Water Supply I: Four Towns. In that activity, students investigate a case (a quadrilateral) in which the perpendicular bisectors are *not* concurrent. Seeing that case first should make it seem more significant to students that perpendicular bisectors are concurrent in a triangle.

INVESTIGATE

1–2. Answers will vary.

A More General Problem

3. Students should observe and conjecture that the perpendicular bisectors of any triangle are always concurrent, although in a quadrilateral they are not. Make sure that they explore a large variety of different triangles to convince themselves of the generality of this result.
4. Students should find as they drag the vertices that the sketch supports the conjecture they made in Question 3.

EXPLAINING

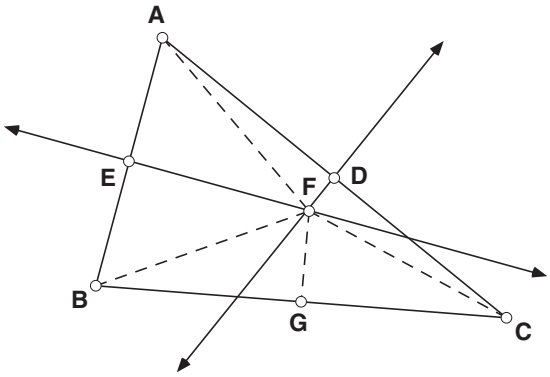
First give students an opportunity to try to explain the result on their own before giving them the Explaining section of the activity.

5. All the points are equidistant from village 1 and village 2.
6. All the points are equidistant from village 2 and village 3.
7. Point *P* is equidistant from village 1 and village 3 and must therefore lie on the perpendicular bisector of the line segment connecting village 1 to village 3.
8. The perpendicular bisector of the third side must also pass through point *P.*

Note: The concurrency of the perpendicular bisectors and the existence of a circumcircle are traditionally proved (explained) by means of congruency as follows. Students may gain insights by comparing this proof with the one based on equidistance in the activity.

Proof

Consider the triangle *ABC* shown here. Construct the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ to intersect in *F.* Join *F* to *G,* the midpoint of $\overline{BC}$. It is now necessary to prove that $m\angle FGB = 90°$—in other words, that $\overline{FG}$ is also the perpendicular bisector of $\overline{BC}$.



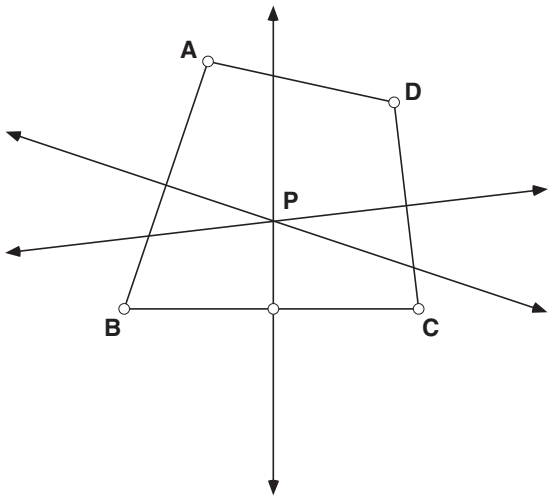
ΔAEF is congruent to ΔBEF (SAS); therefore, $FB = FA$. ΔADF is congruent to ΔCDF (SAS); therefore, $FC = FA$. Thus, $FB = FC$, and it follows that triangles *BGF* and *CGF* are also congruent (SSS). Therefore, $m\angle FGB = m\angle FGC = 90°$.

Further Exploration

1. The circumcenter falls inside if the triangle is acute, on the hypotenuse if it is a right triangle, and outside if it is obtuse.
2. a. The circle now passes through all four vertices.
   b. The fourth perpendicular bisector also passes through the intersection of the other three.
   c. Students should note that if three of the perpendicular bisectors of a quadrilateral are concurrent, the fourth perpendicular bisector is concurrent with the other three and the quadrilateral is cyclic (its four vertices all lie on a circle).

The proof of the result in Further Exploration, Question 2c, is similar to the one for the triangle. In quadrilateral *ABCD,* consider the perpendicular bisectors of sides *AB, BC,* and *CD,* and assume that they are concurrent in point *P.*

© 2012 Key Curriculum Press



All the points on the perpendicular bisector of $\overline{AB}$ are equidistant from points *A* and *B*. Similarly, all the points on the perpendicular bisector of $\overline{BC}$ are equidistant from points *B* and *C*. Also, all the points on the perpendicular bisector of $\overline{CD}$ are equidistant from points *C* and *D*. Therefore, if all the perpendicular bisectors are concurrent in the same point *P*, point *P* must be equidistant from points *A*, *B*, *C*, and *D*. It follows that a circle can be drawn, with point *P* as its center, passing through all four vertices. Since point *P* is equidistant from points *A* and *D*, it must therefore also lie on the perpendicular bisector of $\overline{AD}$.

Similarly, if four perpendicular bisectors of a pentagon are concurrent, the fifth perpendicular bisector is always concurrent with the intersection of the other four, and the pentagon is cyclic. This is now easily generalized to the following: If $n - 1$ perpendicular bisectors of an *n*-gon are concurrent, all of them are (and therefore the *n*-gon is cyclic).

TRIANGLE ANGLE SUM (PAGE 34)

It is still advisable to avoid using the word *proof* and to focus on the explanation of the result rather than on its verification. It should again be emphasized to students that no amount of experimentation provides us with an adequate explanation; for that, we require some form of logical explanation.

It is important not to delay the introduction to logical explanation (proof) too long, because a prolonged purely empirical approach can become so fossilized in students' thinking that it is difficult to change later. Many students do exhibit a need for deeper understanding, and the sooner this habit of mind is cultivated the better.

Although you could let students discover the angle sum result of a triangle simply by measuring each angle and adding them with a calculator, such an approach, according to the van Hiele theory, does not provide an appropriate conceptual structure for the proof (explanation) that is to follow later. The initial empirical activity given here, however, provides the conceptual and visual framework on which the formal explanation (proof) is later built.

Prerequisites: Properties of parallel lines.

Sketch: Triangle Sum.gsp.

CONJECTURE

1. Because of the translations and rotations (and therefore the corresponding congruence of angles), the three angles of the triangle form a straight line. In other words, the sum of the angles of a triangle is always equal to the measure of a straight angle. Students should look at specific triangles as well as limiting cases.
2. The sum of the measures of the angles of a triangle is 180°.

EXPLAINING

3. $m\angle BAC = m\angle ECD$ (corresponding angles).
4. $m\angle ABC = m\angle BCE$ (alternate angles).
5. $m\angle ACB + m\angle BCE + m\angle ECD = 180°$. (*ACD* is a straight line from construction.)

© 2012 Key Curriculum Press