Why Still Factorize Algebraic Expressions by Hand?

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"*Mathematicians have always dreamed of building machines to reduce the drudgery of routine calculations. The less time you spend calculating, the more time you can spend thinking.*" - Ian Stewart (2008, p. 345)

This paper presents and briefly discusses an algebraic expression that came up in a proof that could not be factorized by current computer algebra systems, but had to be done by hand using high school techniques.

(among others)

In De Villiers (2012), the experimental discovery with *Sketchpad* is described ^ of the following interesting inequality for a parallelo-hexagon – a hexagon *ABCDEF* with opposite sides equal and parallel (see Figure 1): $AD^2 + BE^2 + CF^2 \le 4(AB^2 + BC^2 + CD^2)$.

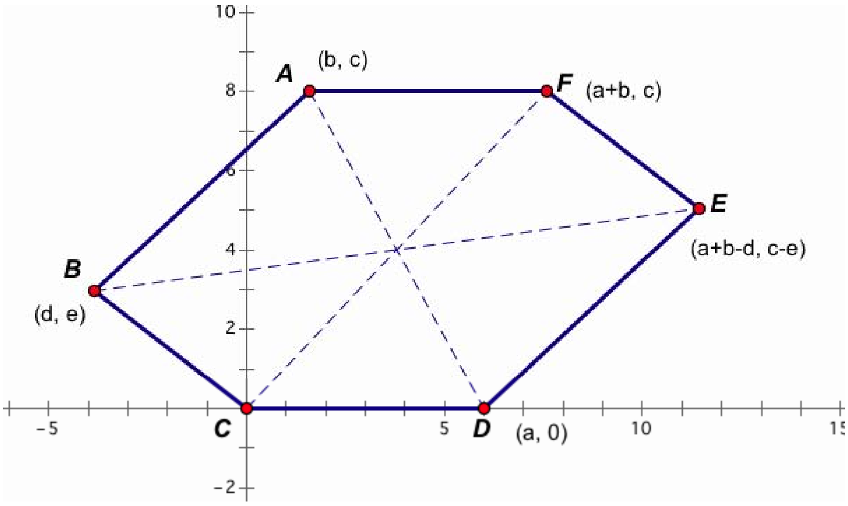

FIGURE 1: Parallelo-hexagon *ABCDEF*.

PROOF

Consider Figure 1 where a parallelo-hexagon is placed on a coordinate grid with vertices C, D, A and B having respective coordinates (0, 0); (a, 0); (b, c) and (d, e). It then follows from the symmetric properties of a parallelo-hexagon that the respective coordinates of vertices E and F are ($a + b - d$, $c - e$) and ($a + b$, c). This gives us the following two equations:

$$4(AB^2 + BC^2 + CD^2) = 4\left[(c-e)^2 + (b-d)^2 + d^2 + e^2 + a^2\right]$$

$$AD^2 + BE^2 + CF^2 = c^2 + (b-a)^2 + (a+b-2d)^2 + (c-2e)^2 + (a+b)^2 + c^2$$

Expanding and subtracting the second equation from the first gives us:

$$\begin{aligned} 4(AB^2 + BC^2 + CD^2) - (AD^2 + BE^2 + CF^2) &= a^2 + b^2 + c^2 + 4d^2 + 4e^2 - 2ab + 4ad - 4bd - 4ce \\ &= (a-b)^2 + 4d(a-b) + 4d^2 + (c-2e)^2 \\ &= (a-b+2d)^2 + (c-2e)^2, \end{aligned}$$

which completes the proof, since the difference of these equations is the sum of two squares, which is always greater than or equal to zero.

Computer Algebra

Computer algebra software packages are becoming widespread, with several freeware options currently available. Such computer algebra software has been available on graphing calculators for some time, and more recently has also started appearing on smart phones. This kind of software is strongly challenging the amount of time traditionally spent teaching and drilling learners at school on how to do complicated algebraic manipulation, since most of this routine work can now be done with the mere press of a button. For example, as shown in Figure 2 using the freeware programme *Eigenmath*, one can simply type in an algebraic expression such as $6x^2+7x-3$ or $6x^3+5x^2-29x-10$ and immediately factorize the expression by pressing the 'Factor' button. In the same way, one can easily expand or multiply out the expression $(4x^2-x+2)(3x^2+5x-3)$ simply by pressing the 'Expand' button.

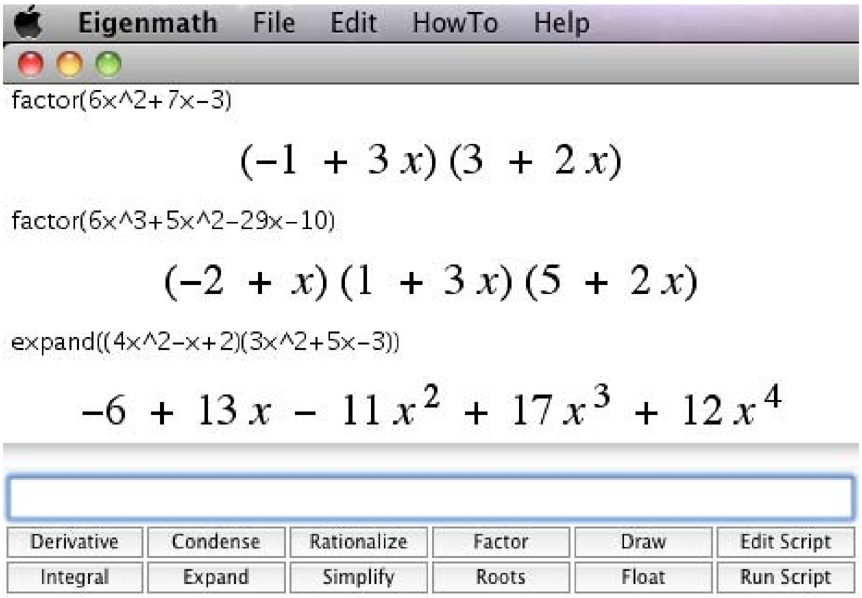

FIGURE 2: Factorizing and expanding algebraic expressions using *Eigenmath*.

As one who does not personally relish tedious algebraic manipulation and usually seeks every opportunity to save time and use available software to perform mundane tasks, I was curious to see how a computer algebra system might be used in the proof of $AD^2+BE^2+CF^2 \le 4(AB^2+BC^2+CD^2)$ for the parallelo-hexagon discussed at the start of this article, rather than me doing all the algebra by hand. The freeware programme *Eigenmath* had no problem expanding and simultaneously simplifying the original equation,

$$4\left[(c-e)^2+(b-d)^2+d^2+e^2+a^2\right]-c^2+(b-a)^2+(a+b-2d)^2+(c-2e)^2+(a+b)^2+c^2$$

correctly (as shown in Figure 3) to:

$$a^2+b^2+c^2+4d^2+4e^2-2ab+4ad-4bd-4ce.$$ (rewritten here in descending powers of a, b, c, d and e)

Disappointingly, however, the software was not able to group and then factorize the equation into the required form of the sum of two squares, simply returning it, as shown in the last line in Figure 3, in the same form as before.

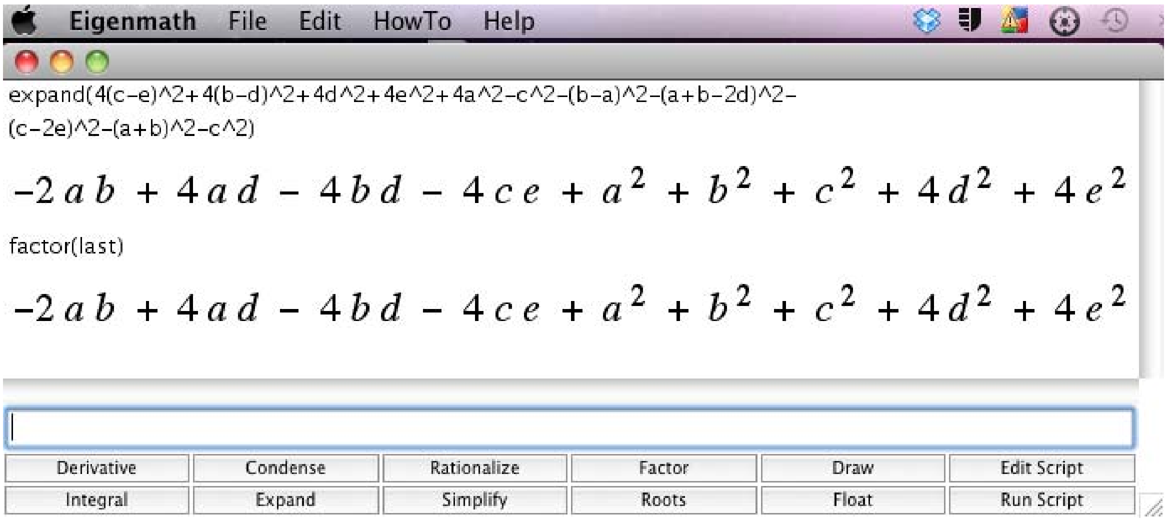

FIGURE 3: Attempting to factorize as the sum of two squares.

Even the computer algebra system *Mathematica,* as powerful as it is, could not group and simplify the expression further. Readers may wish to try for themselves with the online (scaled down) version of *Mathematica* at: http://www.wolframalpha.com/examples/Algebra.html

One can see why the software had difficulty factorizing this expression, since it is NOT routine at all, and involves a number of steps. It firstly involves the grouping and respective factorization of the quadratic expressions $a^2-2ab+b^2$ and $c^2-4ce+4e^2$ into the respective equivalent forms $(a-b)^2$ and $(c-2e)^2$ as well as the taking out of the common term $4d$ to factorize $4ad-4bd$ to $4d(a-b)$. Lastly, it involves recognizing the quadratic expression in $(a-b)$ and d in order to factorize $(a-b)^2+4d(a-b)+4d^2$ to $(a-b+2d)^2$ and thereby complete the proof.

Even though the factorization procedures individually are routine exercises at more or less Grade 10 level, it does require some ingenuity and ability to 'see' and select the appropriate groupings of terms. Actually, for the given proof the original expression was not "factorized", but written as the sum of two squares. So it's no wonder the technology wasn't able to "factorize" the expression - it's simply not possible in this particular instance. At the moment it therefore seems that computer algebra systems are not yet powerful or 'intelligent' enough to handle a non-routine manipulation such as this (even though it can, in the opposite direction, easily "expand" the sum of two squares).

So, this is perhaps an excellent example to demonstrate to one's high school or undergraduate students that a basic facility in algebraic manipulation *by hand* can still come in handy, even in a modern computational age! Perhaps we also need to identify more examples such as these where computing technology fails, or is of little value, and focus more on them rather than the current practice of focusing only on the drill and exercise of routine skills that can be more efficiently handled by computing technology.

REFERENCES

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Stewart, I. (2008). *Taming the infinite*. London: Quercus Publishing.

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