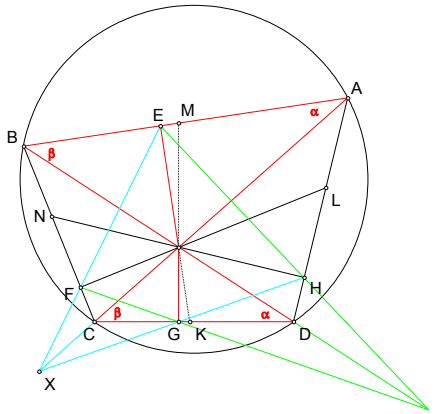
The conic problem

The diagonals of a cyclic quadrilateral *ABCD* meet at *P*. The feet of the perpendiculars from *P* to *AB*, *BC*, *CD*, *DA* are *E*, *F*, *G*, *H*. The lines *EP*, *FP*, *GP*, *HP* meet the opposite sides of *ABCD* at *K*, *L*, *M*, *N*. Prove that *E*, *F*, *G*, *H*, *K*, *L*, *M*, *N* lie on a conic.

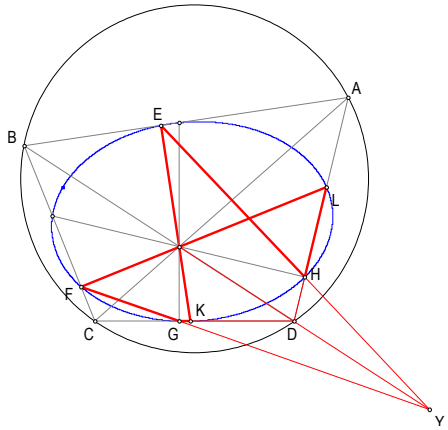
Proof

(1) Triangles *APB*, *DPC* have equal angles and are therefore similar; and *PE*, *PG* are corresponding altitudes. Thus $\frac{AE}{EB}=\frac{DG}{GC}$. Similarly $\frac{BF}{FC}=\frac{AH}{AD}$.

(2) Let the lines *EF, AC* meet at *X*. Then Menelaus' Theorem applied to *ABC* and the line *EFX* gives $\frac{AE}{EB}\times\frac{BF}{FC}\times\frac{CX}{XA}=-1$. Applying (1) transforms this into $\frac{DG}{GC}\times\frac{AH}{HD}\times\frac{CX}{CA}=-1$, so the points *G*, *H*, *X* on the sides of *DCA* are collinear. In other words, *EF*, *GH*, *AC* are concurrent at *X*. Similarly *FG*, *HE*, *BD* are concurrent in a point *Y* say.



(3) Consider the hexagon *FGKEHL*: we find that *FG* · *EH* = *Y*, *GK* · *HL* = *D*, *KE* · *LF* = *P*. These all lie on *BD*, which is therefore a Pascal line. Consequently *E*, *F*, *G*, *H*, *K*, *L* lie on a conic α say. Any conic passing through five of these six points can be identified with α.



Now move each vertex of the hexagon anticlockwise to the corresponding point on the next side. This gives *GHLFEM*, whose opposite sides meet in the collinear points *X*, *A*, *P*. Thus *E*, *F*, *G*, *H*, *L*, *M* lie on a conic, which we identify with α. So *M* lies on α.

Repeating the process gives *HEMGFN*, with a Pascal line *YBP*. Hence α, passing through *E*, *F*, *G*, *H*, *M*, also contains *N*. Thus the eight points lie on a conic.

This proof also holds if *ABCD* is a crossed cyclic quadrilateral.

Michael Fox & Michael de Villiers 16 May 2005