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Viviani's Theorem and Its Extension

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Let $\mathcal{P}$ be a polygon or polyhedron consisting of both boundary and interior points. Define the distance sum function $\mathcal{V} : \mathcal{P} \to \mathbb{R}$, where for each point $P \in \mathcal{P}$ the value $\mathcal{V}(P)$ is defined as the sum of the distances from P to the sides (faces) of $\mathcal{P}$. We say that $\mathcal{P}$ has the *constant Viviani sum* property, abbreviated *CVS*, if the function $\mathcal{V}$ is constant.

Viviani (1622–1703), who was a student and assistant of Galileo, discovered that equilateral triangles have the *CVS* property. His theorem can be easily proved by using areas. Joining a point P inside the triangle to its vertices divides it into three parts. The sum of their areas will be equal to the area of the original one. Therefore, $\mathcal{V}(P)$ will be equal to the height of the triangle and the theorem follows. The importance of Viviani's theorem is indicated by the fact that his teacher Torricelli (1608–1647) used it to locate the Fermat point of a triangle [**2**, p. 443].

What happens for general polygons and polyhedra?

There is a surprising connection between Viviani's theorem and linear programming. This article lays out this connection, which leads to a complete analysis of the behavior of $\mathcal{V}$ for polygons and polyhedra —both concave and convex.

Convex polygons and polyhedra

The case of triangle Given a triangle $\triangle ABC$, let a_1, a_2, a_3 be the lengths of the sides BC, AC, AB, respectively. Let P be a point inside the triangle and let h_1, h_2, h_3 be the distances of P to the three sides (see Figure 1).

For $1 \le i \le 3$, let $x_i = \frac{h_i}{\mathcal{V}(P)}$, where $\mathcal{V}(P) = \sum_{i=1}^{3} h_i$. Clearly, for each $1 \le i \le 3$, we have $0 \le x_i \le 1$ and $\sum_{i=1}^{3} x_i = 1$. Let $\boldsymbol{x} = (x_1, x_2, x_3)$, and consider the linear function in three variables $F(\boldsymbol{x}) = \sum_{i=1}^{3} a_i x_i$. Now, this function is closely related to $\mathcal{V}$. More precisely,

$$F(\boldsymbol{x}) = \sum_{i=1}^{3} a_i x_i = \frac{\sum_{i=1}^{3} a_i h_i}{\sum_{i=1}^{3} h_i} = \frac{2S}{\mathcal{V}(P)},$$

where S is the area of the triangle.

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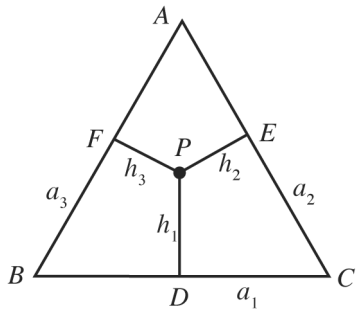

Figure 1. A point P inside the triangle with distances h_1, h_2, h_3.

We have the following linear programming problem: the objective function is

$$F(\boldsymbol{x}) = \sum_{i=1}^{3} a_i x_i,$$

subject to the constraints

$$\begin{cases} \sum_{i=1}^{3} x_i \leq 1 \\ x_i \geq 0, \qquad 1 \leq i \leq 3. \end{cases} \tag{1}$$

Note that, $F(\boldsymbol{x})$ takes equal values in a subset of points of the feasible region (1) if and only if the function $\mathcal{V}$ takes equal values at the corresponding points inside the triangle.

Solving this problem means maximizing or minimizing the objective function in the feasible region, and this optimal value must occur at some corner point. For that, one may use the simplex method, which is easy in this case because the simplex tableau contains only two rows. But, since we are not seeking optimal values, to better understand the connection with Viviani's theorem, we use a geometric method. The feasible region will be the right pyramid with vertex $(0, 0, 1)$ and basis vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and the isoprofit planes are the planes $F(\boldsymbol{x}) = \sum_{i=1}^{3} a_i x_i = c$, where c is constant (see Figure 2).

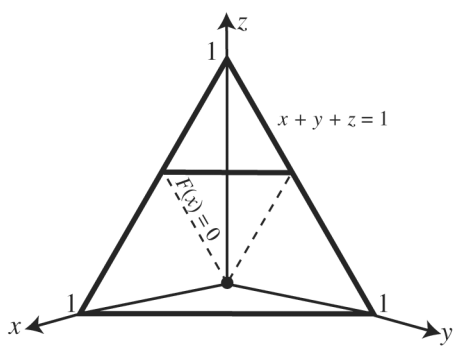

Figure 2. The feasible region (solid) with an isoprofit plane: $F(x) = 0$ (dashed).

These isoprofit planes will meet the face $\sum_{i=1}^{3} x_i = 1$ across a line which in its turn corresponds to a segment inside the triangle $\triangle ABC$. The function $\mathcal{V}$ will be constant

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for all points on this segment, which we call an *isosum* segment. Moreover, moving perpendicular to an isosum segment changes the value of $\mathcal{V}$. Hence, $\mathcal{V}$ does not take the same value at any three non-collinear points inside the triangle.

This is true in general, except for one case: where the isoprofit planes, $\sum_{i=1}^{3} a_i x_i = c$, are parallel to the face $\sum_{i=1}^{3} x_i = 1$. This exceptional case occurs if and only if the vectors (a_1, a_2, a_3) and $(1, 1, 1)$ are linearly dependent, i.e.,

$$(a_1, a_2, a_3) = \lambda(1, 1, 1),$$

which happens if and only if the triangle is equilateral. This proves the following theorem.

Theorem 1. *Any triangle can be divided into parallel segments on which $\mathcal{V}$ is constant. Furthermore, the following conditions are equivalent:*

- *The triangle $\triangle ABC$ has the CVS property.*
- *There are three non-collinear points, inside the triangle, at which $\mathcal{V}$ takes the same value.*
- *$\triangle ABC$ is equilateral.*

Extension to convex polygons We attack the general case with analytic geometry.

Theorem 2.

(a) *Any convex polygon can be divided into parallel line segments, called isosum segments, on which $\mathcal{V}$ is constant.*

(b) *If $\mathcal{V}$ takes equal values at three non-collinear points, inside a convex polygon, then the polygon has the CVS property.*

Proof. Given a polygon with n sides, we embed it in a Cartesian plane. Suppose that its sides lie on lines with equations

$$\alpha_i X + \beta_i Y + \gamma_i = 0. \tag{2}$$

Since the whole polygon lies on one side of the line (2), the expression $\alpha_i x + \beta_i y + \gamma_i$ has one sign for all $P = (x, y)$ inside the polygon. Therefore, the distance, h_i, of the point P from each side of the polygon is given by the equation

$$h_i = (-1)^{\varepsilon_i} \frac{\alpha_i x + \beta_i y + \gamma_i}{\sqrt{\alpha_i^2 + \beta_i^2}},$$

where $\varepsilon_i \in \{0, 1\}$, and the function $\mathcal{V}$ has a linear expression

$$\mathcal{V}(x, y) = \sum_{i=1}^{n} (-1)^{\varepsilon_i} \frac{\alpha_i x + \beta_i y + \gamma_i}{\sqrt{\alpha_i^2 + \beta_i^2}}. \tag{3}$$

Letting this sum equal some constant c, we get the equation of a line:

$$\sum_{i=1}^{n} (-1)^{\varepsilon_i} \frac{\alpha_i x + \beta_i y + \gamma_i}{\sqrt{\alpha_i^2 + \beta_i^2}} = c.$$

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If the point $P = (x, y)$ is restricted to be inside the polygon, then for different values of c, we get parallel isosum segments. On each such segment, the function $\mathcal{V}$ takes the constant value c. This proves part (a).

The equation (3) is independent of (x, y), if and only if the variable part of $\mathcal{V}$ vanishes. But then $\mathcal{V}$ is constant and so the polygon has the *CVS* property.

If there are three non-collinear points inside the polygon at which $\mathcal{V}$ takes the same value, then there exist two different isosum segments on which the function $\mathcal{V}$ takes the same value. This happens if and only if $\mathcal{V}$ is constant, and so once again the polygon has the *CVS* property. This is (b). ■

As a consequence of Theorem 2, we have the following beautiful corollary.

Corollary 1.

(a) *If there is an isometry of the plane which fixes the polygon but not an isosum segment, then the polygon has the CVS property.*

(b) *If a convex polygon possesses a rotational symmetry, then the polygon has the CVS property.*

(c) *If a convex polygon possesses a reflection symmetry across an axis l, then the polygon has the CVS property or otherwise the isosum segments are perpendicular to l.*

Proof.

(a) An isometry of the plane which fixes the polygon but not some isosum segment, assures the existence of three non-collinear points inside the polygon at which $\mathcal{V}$ takes equal values. Thus (a) follows immediately from the theorem.

(b) Follows from (a).

(c) If the isosum segments are not fixed by the reflection, then according to (a) the polygon has the *CVS* property. Otherwise, if fixed, then the isosum segments must be perpendicular to the reflection axis. ■

From the corollary, we deduce that all *regular* polygons have the *CVS* property. Besides, any parallelogram has this property, since it possesses rotational symmetry around its centroid by an angle of 180°.

Obviously, the existence of reflection symmetries by different axes will imply a rotational symmetry, and hence the polygon must have the *CVS* property. Moreover, for triangles and quadrilaterals, the existence of a rotational symmetry characterizes those possessing the *CVS* property, since in these cases the polygons would be only equilateral triangles and parallelograms. Consequently, an n-gon, for $n \geq 5$, that does not possess the *CVS* property must have at most one symmetry which is the reflection symmetry.

Extension to convex polyhedra This is routine. We obtain these results.

Theorem 3.

(a) *Any convex polyhedron can be divided into parallel planar cross sections, called isosum cross sections, on which $\mathcal{V}$ is constant.*

(b) *If $\mathcal{V}$ takes equal values at four non-coplanar points, inside a convex polyhedron, then the polyhedron has the CVS property.*

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Corollary 2.

(a) *If there is an isometry of space which fixes the polyhedron but not an isosum cross section, then the polyhedron has the CVS property.*

(b) *If a convex polyhedron possesses two rotational symmetries, around different axes, then the polyhedron has the CVS property.*

It follows that all regular polyhedra and regular prisms have the *CVS* property. Likewise, any parallelepiped has the *CVS* property, since it possesses three rotational symmetries by an angle of 180°, around axes passing through the centroids of parallel faces.

Concave polygons and polyhedra

The situation for concave polygons and polyhedra is quite different. Theorems 2 and 3 are no longer valid. Moreover, concave polygons and polyhedra never have the *CVS* property. Surprisingly, with a little more effort, we can generalize Theorems 2(a) and 3(a). Concerning polygons, the crucial difference is that the points inside a concave polygon no longer lie on the same side of each boundary line.

An example will illustrate. Let *ABCD* be the concave kite with vertices $(0, 8)$, $(-6, 0)$, $(0, 2.5)$, $(6, 0)$, respectively. Let l_1, l_2, l_3, l_4 be the lines containing the sides *AB*, *BC*, *CD*, *DA* and E, F be the intersection points of l_1, l_3 and l_2, l_4, respectively (see Figure 3). Then the kite is divided into three distinct *convex* polygonal regions, namely; *AECF*, *EBC*, and *FCD*. Within each of these regions an expression may be found for $\mathcal{V}$ like (3) and the reasoning used to prove Theorem 2 applied. Therefore each region can be divided into parallel isosum segments, but in different directions (see Figure 3).

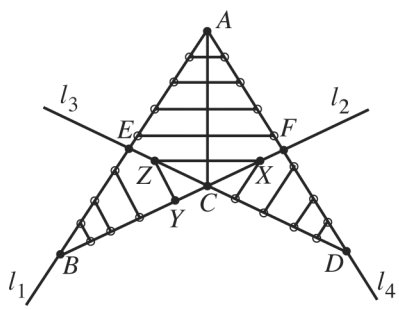

Figure 3. The isosum segments have different directions and X, Y, Z are non-collinear points with equal distance sum from the sides of *ABCD*.

Note that, two isosum segments from neighboring regions which meet at a point on the common edge, define three non-collinear points with the same distance sum from the sides of *ABCD*.

These ideas can be generalized to any concave polygon or polyhedron.

Theorem 4.

(a) *Any concave polygon can be divided into convex polygonal regions which can be divided into parallel isosum segments. Isosum segments of neighboring regions have different directions.*

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(b) *There are always three non-collinear points inside a concave polygon which have equal distance sum from the sides.*

(c) *Any concave polyhedron can be divided into convex polyhedral regions which can be divided into parallel isosum cross sections. Isosum cross sections of neighboring regions have different directions.*

(d) *There are four non-coplanar points inside a concave polyhedron which have equal distance sum from the faces.*

(e) *Concave polygons and polyhedra never possess the CVS property.*

Proof. We outline the proof for polygons.

(a) Let $\mathcal{P}$ a concave polygon. Extend the sides and find all intersection points of the boundary lines in $\mathcal{P}$. (Figure 4 gives an example.) In this way one partitions $\mathcal{P}$ into convex polygonal regions $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m$. Note that each side of any $\mathcal{P}_j$ lies on a boundary line of $\mathcal{P}$. The points inside any two neighboring regions $\mathcal{P}_i, \mathcal{P}_j$ with a common edge LM, lie on the same side of each boundary line of $\mathcal{P}$ *except* the boundary line l_{LM} containing LM. We claim that the union of these two neighboring regions $\mathcal{P}_i \cup \mathcal{P}_j$, is convex. This is true since the two sides of $\mathcal{P}_i$ and $\mathcal{P}_j$ which meet at L (or M) lie on the same boundary line of $\mathcal{P}$, by construction. (In Figure 4, the points K, L, D are collinear, likewise I, M, D are collinear). Thus all points of $\mathcal{P}_i \cup \mathcal{P}_j$ lie on the same side of the boundary lines of $\mathcal{P}$ except l_{LM}.

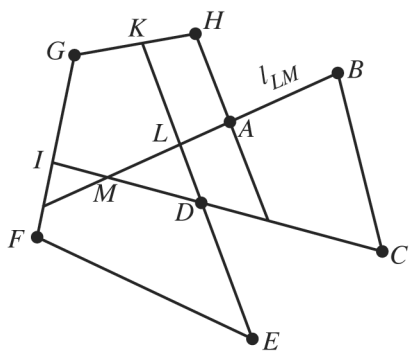

Figure 4. Partitioning the concave polygon *ABCDEFGH* into convex polygonal regions.

Each convex polygonal region has parallel isosum segments according to Theorem 2. Isosum segments of neighboring regions $\mathcal{P}_i, \mathcal{P}_j$ have different directions since the linear expression of $\mathcal{V}$ on these regions differ in *one* sign.

(b) Take any point P on a common edge of two neighboring regions $\mathcal{P}_i, \mathcal{P}_j$ and let s_1, s_2 be two isosum segments of $\mathcal{P}_i, \mathcal{P}_j$, respectively, issuing from P. Let Q_1, Q_2 be two points on s_1, s_2, respectively. Since s_1, s_2 have different directions then P, Q_1, Q_2 are non-collinear and have the same distance sum from the sides.

(e) The distance sum function $\mathcal{V}$ on two neighboring regions $\mathcal{P}_i, \mathcal{P}_j$ has the same linear expressions with the same signs except one. Thus $\mathcal{V}$ cannot be constant on $\mathcal{P}$. ■

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Examples

Example 1 (Kite). Consider a kite in the Cartesian plane with vertices, $A = (0, \beta)$, $B = (-\alpha, 0)$, $C = (0, \gamma)$, $D = (\alpha, 0)$, where $\alpha, \beta > 0$ and $\gamma < 0$.

The equations of the lines containing the sides AB, BC, CD, DA are

$$\begin{aligned} -\alpha y + \beta x + \alpha\beta &= 0, \\ -\alpha y + \gamma x + \alpha\gamma &= 0, \\ \alpha y + \gamma x - \alpha\gamma &= 0, \\ \alpha y + \beta x - \alpha\beta &= 0. \end{aligned}$$

Thus $\mathcal{V}$ is given by

$$\mathcal{V}_{ABCD} = \frac{-\alpha y + \beta x + \alpha\beta}{\sqrt{\alpha^2 + \beta^2}} - \frac{-\alpha y + \gamma x + \alpha\gamma}{\sqrt{\alpha^2 + \gamma^2}} + \frac{\alpha y + \gamma x - \alpha\gamma}{\sqrt{\alpha^2 + \gamma^2}} - \frac{\alpha y + \beta x - \alpha\beta}{\sqrt{\alpha^2 + \beta^2}}.$$

Equivalently,

$$\mathcal{V}_{ABCD} = \frac{-2\alpha y}{\sqrt{\alpha^2 + \beta^2}} + \frac{2\alpha\beta}{\sqrt{\alpha^2 + \beta^2}} + \frac{2\alpha y}{\sqrt{\alpha^2 + \gamma^2}} - \frac{2\alpha\gamma}{\sqrt{\alpha^2 + \gamma^2}}. \tag{4}$$

As a result, the isosum segments have simple equations, namely, $y = c$.

Obviously, equation (4), shows that $\mathcal{V}_{ABCD}$ is independent of variables if and only if $\beta = \gamma$. This means that the polygon is in fact a parallelogram and hence has the *CVS* property. On the other side, if $\beta \neq \gamma$, then the kite does not have the *CVS* property, but can be divided into isosum segments which are parallel to the diagonal BD, consistent with Corollary 1(c).

Example 2 (Quadrilateral). Consider the quadrilateral with vertices (0, 0), (3, 0), (1, 2), (0, 1). Direct computations show that the isosum segments are parallel to the line $y = (1 + \sqrt{2})x$ (see Figure 5), so this quadrilateral does not have the *CVS* property.

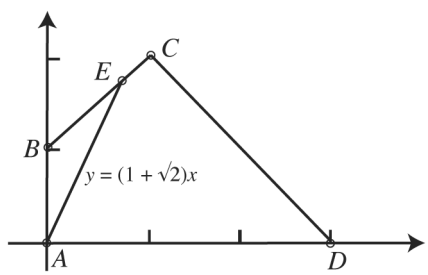

Figure 5. The isosum segments are parallel to the line $y = (1 + \sqrt{2})x$.

Example 3 (Asymmetric pentagon). A pentagon $ABCDE$ with angles 70°, 110°, 130°, 60°, and 170°, although it has no symmetry, possesses the *CVS* property, since it is composed of a parallelogram and an equilateral triangle.

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Example 4 (Pyramid with a rotational symmetry). Consider the rectangular pyramid with vertices $A = (0, 0, \alpha)$, $B = (\beta, 0, 0)$, $C = (0, \gamma, 0)$, $D = (-\beta, 0, 0)$, and $E = (0, -\gamma, 0)$, where $\alpha, \beta, \gamma > 0$. The faces ABC, ACD, ADE, and AEB lie on the following planes:

$$\begin{aligned}
\alpha\gamma x + \alpha\beta y + \beta\gamma z &= \alpha\beta\gamma,\\
-\alpha\gamma x + \alpha\beta y + \beta\gamma z &= \alpha\beta\gamma,\\
-\alpha\gamma x - \alpha\beta y + \beta\gamma z &= \alpha\beta\gamma,\\
\alpha\gamma x - \alpha\beta y + \beta\gamma z &= \alpha\beta\gamma.
\end{aligned}$$

Letting $\Delta = \sqrt{\alpha^2\gamma^2 + \alpha^2\beta^2 + \beta^2\gamma^2}$ then the altitudes from the point $P = (x, y, z)$ to the faces are given by

$$\begin{aligned}
h_{BCDE} &= z,\\
h_{ABC} &= \frac{-1}{\Delta}(\alpha\gamma x + \alpha\beta y + \beta\gamma z - \alpha\beta\gamma),\\
h_{ACD} &= \frac{-1}{\Delta}(-\alpha\gamma x + \alpha\beta y + \beta\gamma z - \alpha\beta\gamma),\\
h_{ADE} &= \frac{-1}{\Delta}(-\alpha\gamma x - \alpha\beta y + \beta\gamma z - \alpha\beta\gamma),\\
h_{AEB} &= \frac{-1}{\Delta}(\alpha\gamma x - \alpha\beta y + \beta\gamma z - \alpha\beta\gamma).
\end{aligned}$$

Thus

$$\mathcal{V}_{ABCDE} = \left(1 - \frac{4\beta\gamma}{\Delta}\right) z + \frac{4\alpha\beta\gamma}{\Delta}.$$

Evidently, the isosum layers are parallel to the base of the pyramid and it has the *CVS* property if and only if

$$\alpha = \frac{\sqrt{15}\beta\gamma}{\sqrt{\beta^2 + \gamma^2}}.$$

Figure 6 illustrates the special case $\beta = \gamma = 1$ and $\alpha = \sqrt{\frac{15}{2}}$.

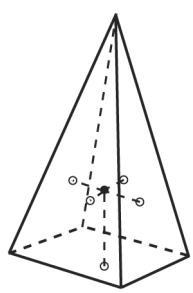

Figure 6. A square pyramid which has the *CVS* property.

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Concluding remarks

Samelson [**5**, p. 225] gave a proof of Viviani's theorem that uses vectors and Chen & Liang [**1**, p. 390–391] used this vector method to prove a converse: if inside a triangle there is a circular region in which $\mathcal{V}$ is constant then the triangle is equilateral.

Kawasaki [**3**, p. 213], with a proof without words, used only rotations to establish Viviani's theorem. There is an extension of the theorem to all regular polygons, by the area method: all regular polygons have the *CVS* property. There is also an extension of the theorem to regular polyhedrons, by a volume argument: all regular polyhedra have the *CVS* property. Kawasaki, Yagi, and Yanagawa [**4**, p. 283] gave a different proof for the regular tetrahedron.

The property of possessing a symmetry is not necessary for the *CVS* property. Example 3 gives an asymmetric polygon with the *CVS* property.

Corollary 1(c) can be extended to polyhedra, but the property of being perpendicular to the reflection plane would not uniquely determine the isosum cross sections.

It is possible to give an algebraic necessary and sufficient condition for the *CVS* property, using the expression of $\mathcal{V}$ given in (3). But a geometric one is better. Thus the substantial question is how can one characterize, geometrically, all polygons and polyhedra that satisfy the CVS property?

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Summary. Viviani's theorem states that the sum of distances from any point inside an equilateral triangle to its sides is constant. Here, in an extension of this result, we show, using linear programming, that any convex polygon can be divided into parallel line segments on which the sum of the distances to the sides of the polygon is constant.

Let us say that a polygon has the *CVS property* if the sum of distances from *any* inner point to its sides is constant. An amazing converse of Viviani's theorem is deduced: if just three noncollinear points inside a convex polygon have equal sums of distances then the polygon has the CVS property. For concave polygons the situation is quite different. For polyhedra analogous results are deduced.

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