Chapter 2

Defining and Classifying

The Oxford Dictionary (1974) describes the verbs "*define*" and "*classify*" respectively as "*state precisely the meaning of words or concepts*" and "*arrange in classes or groups.*" In A Dictionary of Mathematics by McDowell (1961) a "*definition*" is described as "*the explanation of the meaning of a word or phrase. A short description of something by pointing out its properties.*"

A common, but false conception is that there is only one (correct) definition for each defined object in Mathematics; in fact, several different (correct) definitions may exist. A further misconception is that mathematics always starts with definitions - indeed no definitions of mathematical objects are given *a priori* in Nature. Nor do definitions exist independently of human experience in some "*ideal*" Platonistic world, so that all we can do is to "*discover*" them. To the contrary, they are not discoveries, but human "*inventions*" for the main purpose of accurate mathematical communication. We can say that a definition is a mutual agreement amongst interested parties about what a specific object really is or what is meant by certain terminology.

The construction of definitions (defining) is a mathematical activity of no less importance than to make deductions from given definitions. In mathematics we can distinguish between two different types of defining of concepts, namely, **descriptive** (*a posteriori)* and **constructive** (*a priori*) defining (e.g. compare Krygowska, 1971; Freudenthal, 1973:461; Human, 1978:164-165; De Villiers, 1986).

Descriptive defining

"... the describing definition ... outlines a known object by singling out a few characteristic properties". - Hans Freudenthal (1973 : 458)

With the descriptive (*a posteriori*) defining of a concept is meant here that the concept and its properties have already been known for some time and is defined only afterwards. *A posteriori* defining is usually accomplished by selecting an appropriate subset of the total set of properties of the concept from which all the other properties can be deduced. This subset then serves as the definition and the other remaining properties are then logically derived from it as theorems. This process is briefly summarised in Figure 12.

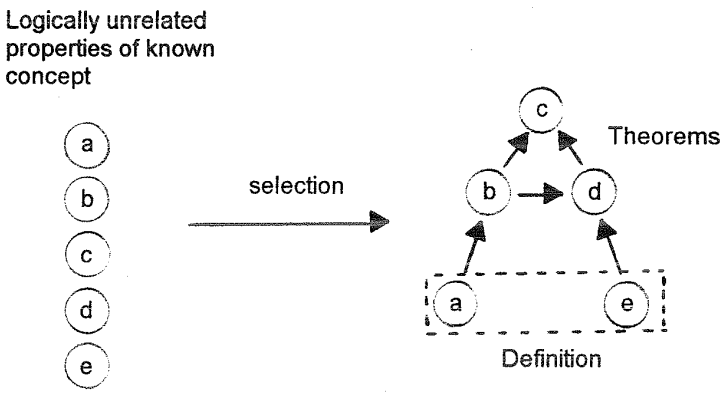


Figure 12

Let's for example consider the *a posteriori* defining of a kite, assuming that we have already discovered and formulated the following properties of it:

(a) two pairs of equal adjacent sides (or two isosceles triangles with a common base)
(b) perpendicular diagonals
(c) one diagonal is bisected by the other
(d) one pair of opposite angles is bisected by a diagonal
(e) one diagonal (the one in (d)) is a line of symmetry
(f) one pair of equal opposite angles

Before proceeding any further, first formulate one or two of your own definitions for a kite. (Check your definitions by construction and/or proof).

To define a kite we could of course list all the above properties, but it is common practice to keep a definition as short as possible. In other words, a good definition avoids unnecessary information, i.e. it must be *economical.* We therefore usually do not include in the definition *all* the properties of a set of objects being defined, but only **necessary** properties to ensure that we obtain the elements of that set. However, more serious than including too much information, is to include too little, in which case there are objects which comply with the definition, but are not elements of the set one wants to define. In other words, for a definition to be correct, it must contain **sufficient** properties to ensure that not only do we obtain the elements of the set we want to define, but only those elements (and not any others).

Accurately **construct** quadrilaterals which comply to each of the conditions in the definitions for kites in (a), (b) and (c) below. (Use ruler, compass and protracter, or if available, computer programmes such as *Sketchpad* or *Cabri-geometre*).

(a) A kite is a quadrilateral with perpendicular diagonals.

(b) A kite is a quadrilateral with two pairs of equal adjacent sides and one pair of opposite angles equal.

(c) A kite is a quadrilateral with perpendicular diagonals with one being bisected by the other.

Now carefully consider the following questions:

(i) Is the given information *sufficient* for the construction of a kite?

(ii) Did you use *all* the information given in the definition to make your construction? If not, which information did you use? Which information did you not use and why not?

The first one is not a correct definition of a kite, since it does not contain sufficient properties. For example, we can construct a diagonal and another one perpendicular on it, and then connecting the endpoints to obtain a quadrilateral as shown in Figure 13 which is clearly not a kite. Note that the statement *"a kite is a quadrilateral with perpendicular diagonals"* is a *correct statement* about a property of kites BUT that it contains too little information to be used as a **definition.** We would therefore say that *"perpendicular diagonals"* is a necessary, but not sufficient condition for kites.

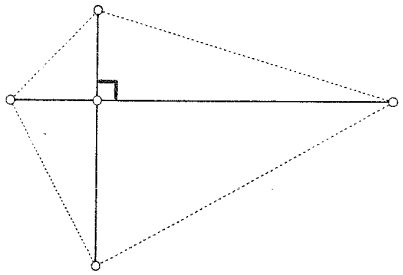


Figure 13

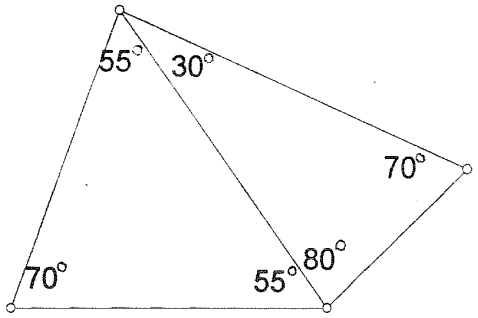


Figure 14

The second one is correct, but uneconomical as it contains too much information. In other words, the conditions are sufficient, but not all of them are necessary. But which can be left out? If we leave out *"two pairs of equal adjacent sides"* we obtain an incorrect definition, since it is possible to construct a quadrilateral with one pair of equal opposite angles which is not a kite (see Figure 14). On the other hand, we can construct a kite according to the condition *"two pairs of equal adjacent sides"* by placing a compass first at A and then at C and drawing circular arcs as shown in Figure 15. In addition, we can easily show that this condition logically implies that *"one pair of opposite angles are equal."* For example, triangles ACB and ACD are clearly congruent (s,s,s) and therefore $\angle B = \angle D$.

The third one is a correct, economical definition as no superfluous information is supplied. In other words, we cannot leave out any of the properties, and we can construct a kite from the

given conditions by first constructing two perpendicular diagonals, then placing a compass at O to chop off equal distances on the one diagonal and connecting the endpoints as shown in Figure 16. In addition, we can show that the other properties of a kite can be logically deduced from the given conditions. For example, in Figure 16 triangles AOB and AOD are clearly congruent $(s, \angle, s)$ and therefore AB=AD. In the same way it can be shown that CB=CD.

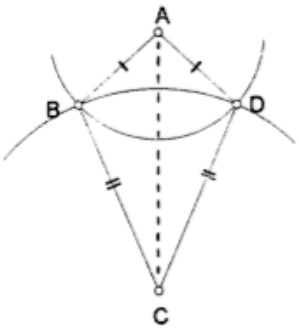


Figure 15

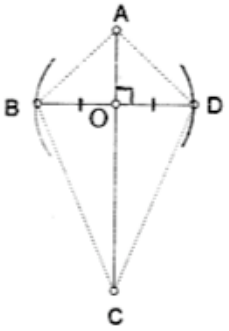


Figure 16

Definitions may also be deficient in other respects than the two aspects mentioned earlier. Critically evaluate each of the definitions for a kite below (that is, say where the mistake lies):

(a) The kites are the set of quadrilaterals with equal diagonals.

(b) The kites are the set of quadrilaterals that are shaped like a kite.

(c) The kites are the set of quadrilaterals with adjacent sides equal.

The first one contains an *incorrect property* since kites in general do not have equal diagonals. The second one is *circular*; it is unacceptable to define an object in terms of itself as it does not explain what that object is. The last one is *ambiguous* (confusing) since a quadrilateral has four pairs of adjacent sides and if *all* pairs of adjacent sides are equal we would have all sides equal, i.e. a rhombus.

Different definitions of the same concept often differ with respect to the proofs required to deduce the other properties not contained in the definition. For example, one definition of a concept may be considered more *economical* than another if the former gives rise to simpler proofs of the other properties. Compare the following three definitions for kites by deducing the other properties from it:

(a) A kite is a quadrilateral with two pairs of adjacent sides equal.

(b) A kite is a quadrilateral with one pair of opposite angles bisected by a diagonal.

(c) A kite is a quadrilateral with one diagonal a line of symmetry.

In the first one we can as before prove triangles ABC and ADC congruent (s, s, s); therefore $\angle B = \angle D$ and diagonal AC bisects the opposite angles A and C (Figure 17a). If we connect B

to D then we can show triangles ABO and ADO congruent $(s,\angle,s)$, from which follows that $\angle AOB = \angle AOD = 90°$ and BO=DO; therefore the diagonals are perpendicular and the one diagonal BD is bisected. From the definition of line symmetry, it therefore now also follows that AC is a line of symmetry.

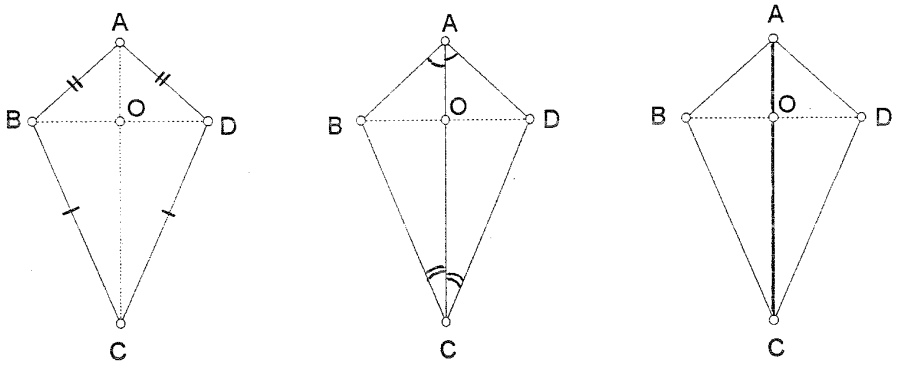


Figure 17

In the second one we similarly have to start out by proving triangles ABC and ADC congruent $(\angle,\angle,s)$, and then proceeding in the same manner as the first, the other properties can be deduced (Figure 17b). In other words, there is no real gain in terms of deductive economy in comparison to the first.

However, the third definition is deductively much more economical. For example, from the definition of line symmetry all the other properties *immediately* follow, i.e. AB=AD, BC=DC, BD⊥AC, BO=DO and AC bisects angles A and C (Figure 17c). It therefore speaks for itself why we would generally prefer the latter definition.

Constructive defining

"... the algorithmically constructive and creative definition ... models new objects out of familiar ones" - Hans Freudenthal (1973 : 458).

Constructive *(a priori)* defining takes places when a given definition of a concept is changed through the exclusion, generalization, specialization, replacement or addition of properties to the definition, so that a new concept is constructed in the process. In other words, a new concept is defined "*into being*", the further properties of which can then be experimentally or logically explored. This process is briefly illustrated in Figure 18. Whereas the main purpose or function of *a posteriori* defining is that of the ***systematization*** of existing knowledge, the main function of *a priori* defining is therefore clearly the production of ***new*** knowledge.

Let's again consider a kite and the definition for it as a quadrilateral with a line of symmetry

through one pair of opposite angles. By the addition of the further constraint that it must have **two** such lines of symmetry, we clearly obtain a rhombus. Similarly, if we add to the definition of a kite the constraint that it must also be cyclic we obtain a *right kite* (see Figure 19). Furthermore, as discussed in the previous chapter we can easily show that it must then have at least one pair of opposite right angles.

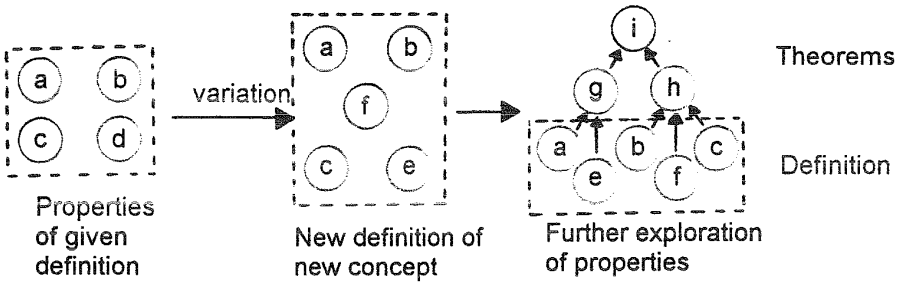


Figure 18

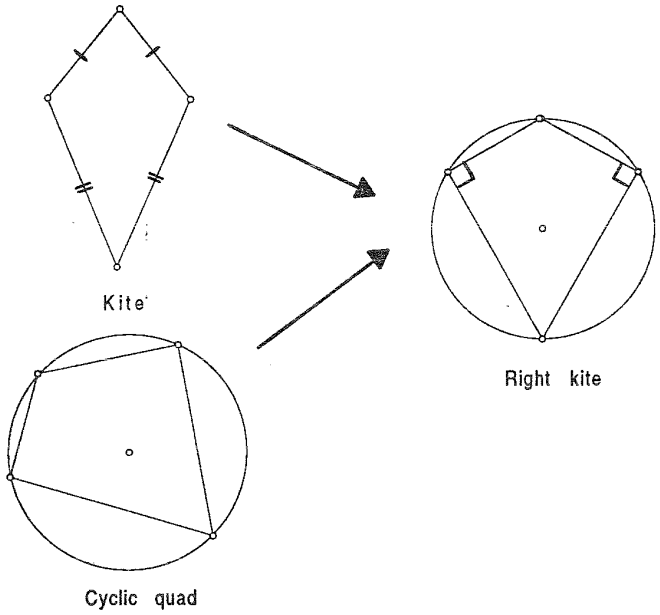


Figure 19

By retaining in turn only one property (condition) of a kite while relaxing all the other conditions, we can *a priori* define four different generalizations of a kite as shown in Figure 20. (The formulation of the definitions for *skew kites, perpendicular, bisecting* and *angle quads* are self-evident from the figure and is left to the reader). We can now explore and hopefully prove other properties of these newly defined figures (see *Questions and Problems 2* later on).

Another way of generalization is to consider the generalization of kites to hexagons, octagons, decagons; i.e. to $2n$-gons in general with $n \geq 2$. In Figure 21 two possible generalizations to

adjacent 2n-gons and *kite 2n-gons* are illustrated. In the former we have *n* pairs of adjacent sides equal and in the latter the property of a diagonal symmetry is maintained. As before, we can now explore and hopefully prove other properties of these newly defined figures (see *Questions and Problems 2* later on).

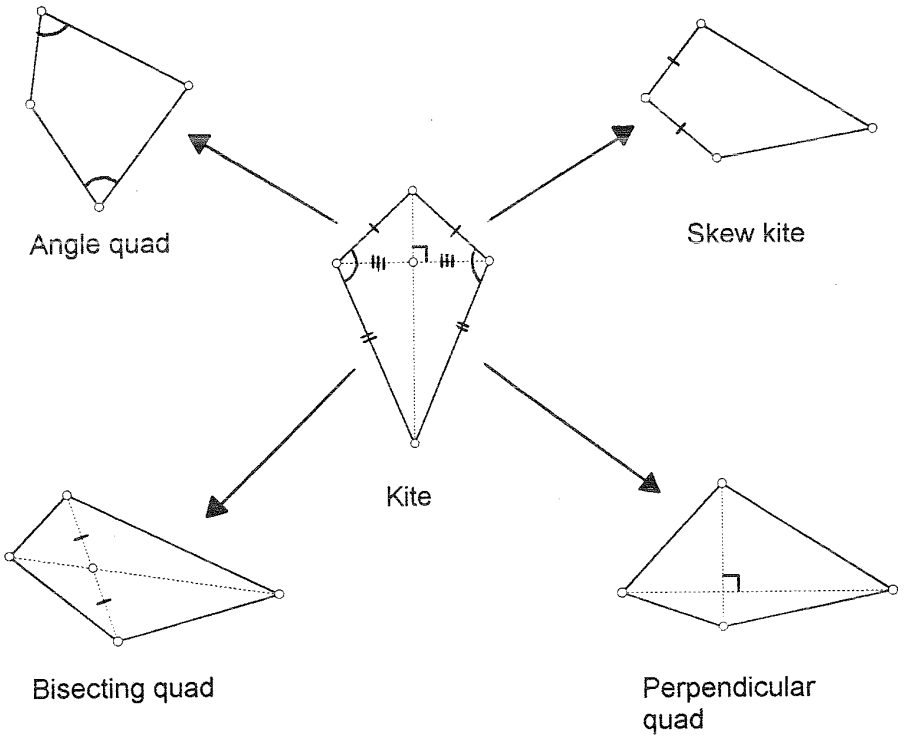


Figure 20

The relationship between classifying and defining

The classification of any set of concepts does not take place independent of the process of defining, but implicitly (or explicitly) involves defining the concepts involved, and vice versa, defining concepts in a certain way automatically also involves their classification. For example, defining a trapezium as "*a quadrilateral with **at least** one pair of opposite sides parallel*" would then imply that a parallelogram is classified as a special trapezium. On the other hand, if a trapezium is defined as "*a quadrilateral with **one** one pair of opposite sides parallel*" the parallelograms are clearly excluded from the trapezia.

We will call a definition like the first a **hierarchical** definition since it allows the inclusion of more particular concepts as subsets of the more general concept being defined. The latter we will call a **partition** definition as the concepts involved are considered to be disjoint from one another. A common misconception is that a partition definition (and classification) is mathematically "*wrong*" simply because it is partitional. However, this is not the case, since a partition definition would be quite correct, provided of course it contains sufficient information

to ensure that all non-examples are excluded. For example, the following partition definition of a parallelogram which excludes rectangles, rhombi and squares may be unconventional, but it is definitely not wrong:
"A parallelogram is a quadrilateral with opposite sides parallel, but not all angles or sides equal".

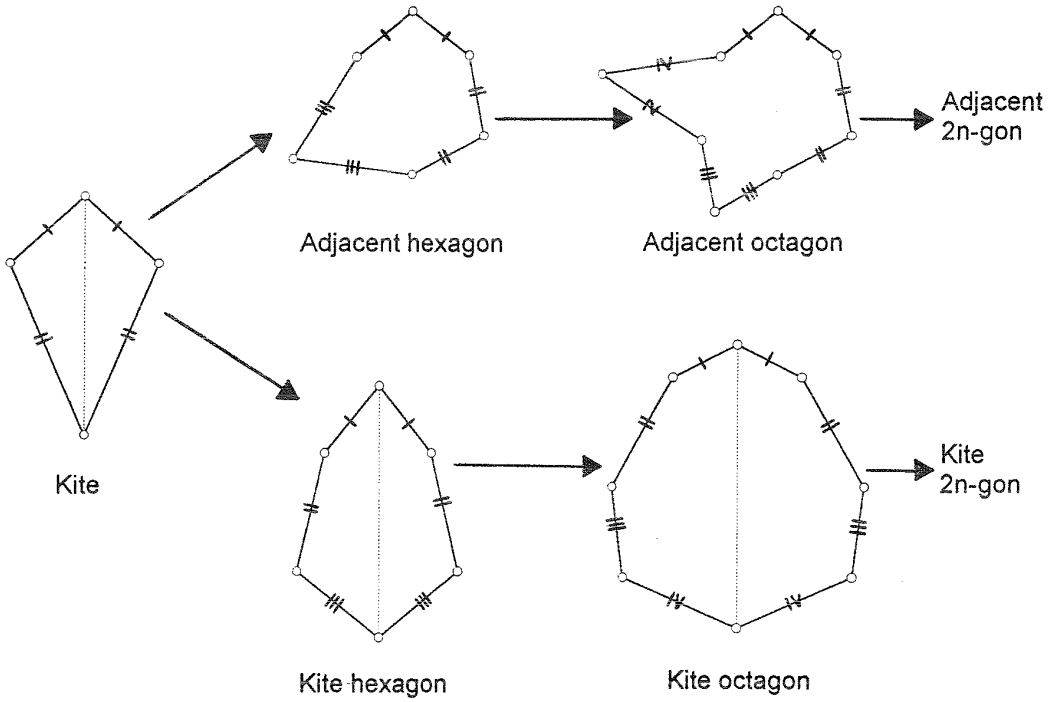


Figure 21

In fact, it is a *correct economical* (partition) definition as it contains only necessary and sufficient properties. An example of a correct, uneconomical partition definition is the following:
"A parallelogram is a quadrilateral with opposite sides equal and parallel, opposite angles equal, diagonals of different length bisecting each other, but not perpendicularly".

This partition definition clearly contains superflous (redundant) information and could be made economical in a number of different ways. An incorrect partition definition on the other hand contains insufficient or incorrect information like the following:
"A parallelogram is a quadrilateral with one pair of opposite sides parallel, but not all angles or sides equal".
"A parallelogram is a cyclic quadrilateral with one pair of opposite sides equal, but not all angles or sides equal".

Sometimes a partition classification and its corresponding definitions are necessary to clearly distinguish between concepts. For example, consider the partition classification of convex,

concave and crossed quadrilaterals shown in Figure 22 with the following possible definitions for the concepts involved.

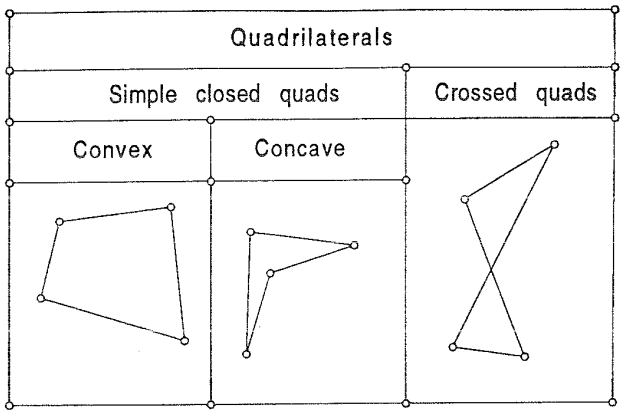


Figure 22

"A quadrilateral is any closed four sided figure in the plane with four vertices".
"A simple closed quadrilateral is a quadrilateral with sides only meeting at the vertices".
"A crossed quadrilateral is a quadrilateral with two of the sides crossing each other at a point other than the vertices".
"A convex quadrilateral is a simple closed quadrilateral with none of its angles reflexive".
"A concave quadrilateral is a simple closed quadrilateral with one of its angles reflexive".

Similarly, it is useful and necessary to partition the kites into convex and concave ones (see Figure 23). It is left as an exercise to verify that a concave kite has all the properties of a convex kite, except that one of its angles is reflexive and one diagonal falls outside the figure.

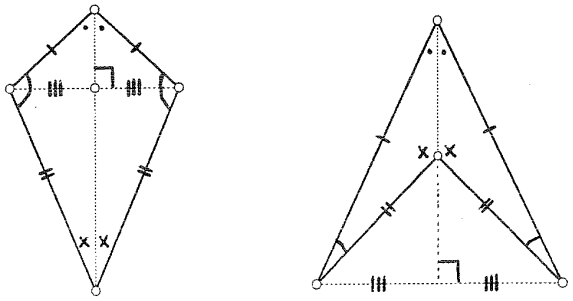


Figure 23

Partitioning is in fact a generally accepted mathematical method in many areas of mathematics, but particularly in the study of topological surfaces and spaces where the fundamental problem is the subdivision of these surfaces and spaces into different disjoint types (e.g. compare

Patterson, 1956). Since a classification and its corresponding definitions are (to a certain degree) arbitrary and not absolute, it is important to realize that the choice between a hierarchical and a partition classification is therefore often a matter of personal choice and convenience.

The fundamental question is therefore: why do we (conventionally prefer a hierarchical classification of the various quadrilaterals rather than a partition classification? Or phrased differently, what advantages does a hierarchical classification in this instance have over a partition classification?

The role and function of a hierarchical classification

Before going any further you are now requested to first complete the following exercise:

(a) Formulate a hierarchical and a partition definition for an isosceles trapezium and critically compare them. Which one is shorter, more economical?

(b) Reformulate the following well-known result from a partition perspective:
"If the midpoints E,F,G and H of the adjacent sides of any quadrilateral are consecutively connected, then EFGH is a parallelogram."

(c)(i) If we classify a rectangle as a parallelogram, is it necessary to prove that the diagonals of a rectangle bisect each other?

(ii) If we partition the rectangles from the parallelograms is it necessary to prove that the diagonals of a rectangle bisect each other?

Some of the most important functions of a hierarchical classification as indicated above are:

* it leads to more economical definitions of concepts and formulation of theorems
* it simplifies the deductive systematization and derivation of the properties of more special concepts
* it often provides a useful conceptual schema during problem solving
* it sometimes suggests alternative definitions and new propositions
* it provides a useful global perspective

Each of these will now be discussed in slightly more detail.

Economical definitions and formulations of theorems

Economy of definition and of formulation of theorems is probably one of the most important advantages of a hierarchical classification. For example, a conventional hierarchical definition for a paralellogram is shorter than the partition definition provided earlier in this section, since

the latter has to include additional properties to exclude the rectangle, rhombi and squares. Also consider the defining of an isosceles trapezium. A hierarchical definition which includes the rectangles, trilateral trapezia and squares would be to say that it is any quadrilateral with an axis of symmetry through (at least) one pair of opposite sides. A partition definition on the other hand, which excludes the rectangles, trilateral trapezia and squares, would have to include the additional conditions that it may not have a right angle nor three sides equal.

A partition classification often also makes the formulation of certain theorems clumsy and cumbersome. Consider for example the following two formulations of well-known results from a partition perspective:
"*If the midpoints E,F,G and H of the adjacent sides of any quadrilateral are consecutively connected, then EFGH is a parallelogram, rectangle, rhombus or square.*"
"*The exterior angle of a cyclic quadrilateral, isosceles trapezium, right kite, trilateral trapezium, rectangle or square is equal to the opposite interior angle.*"

Simplification of deductive structure
By classifying (and defining) a concept A as a subset (special case) of a concept B, it becomes unneccessary to repeat any of the proofs of the properties of concept B for concept A, as they are automatically implied for A by the hierarchical inclusion. For example by classifying a rectangle as a parallelogram, all the theorems which have already been proved for parallelograms are immediately made applicable to the rectangles (and squares). In other words, it is unnecessary to prove for instance that the diagonals of a rectangle (and square) bisect each other or that their opposite sides are equal.

In contrast, if the rectangles were to be excluded from the parallelograms, one would strictly speaking have to again prove that the above properties are also true from the chosen definition for rectangles, whatever it may be (e.g. suppose "*a quadrilateral with four equal angles, but not all sides equal*"). In addition, if the squares are also excluded from the rectangles by a partition definition like the preceding one, the proofs of these properties would have to be repeated again for the squares.

Similarly by hierarchically classifying squares as rhombi and rhombi as kites, it is unnecessary to again prove for instance that the diagonals of a square or rhombus are perpendicular, since this is an easily proved property of the kites. In contrast, a partition classification would require one to repeat the proofs for squares and rhombi. Another good example is that if we hierarchically classify rectangles and squares as isosceles trapezia, it is unnecessary to prove that they have equal diagonals, as this is also an easily proved property of the isosceles trapezia. Apart from the economy of definition or formulation, a hierarchical classification therefore also results in an economical deductive system.

A useful conceptual schema during problem solving

A hierarchical class inclusion is often also useful during problem solving; in particular for proving riders. For example, suppose one wants to prove that a kite with one pair of opposite sides parallel is a rhombus. Using the fact that the rhombi are the **intersection** of the kites and parallelograms, it is **sufficient** therefore to prove that the figure is a parallelogram, since any kite with both pairs of opposite sides parallel must be a rhombus. Other examples can be found in *Questions and problems 1* and will also presented later on.

Alternative definitions and new propositions

Consideration of the hierarchical relationships between concepts may sometimes suggest alternative definitions and new propositions. If for instance concept A is the intersection for two other concepts B and C, then it must obviously possess all the properties of both concepts B and C. By now considering different combinations of the properties of concepts B and C, alternative definitions for concept A, or new propositions, may be suggested.

For example, since any isosceles trapezium is cyclic, and its diagonals are equal, the following alternative definition for isosceles trapezia is possibly suggested:
"An isosceles trapezium is a cyclic quadrilateral with equal diagonals".

Alternatively, it can be formulated as a proposition:
"If a cyclic quadrilateral has equal diagonals then it is an isosceles trapezium".

The keeping in mind of a hierarchical classification can also sometimes enable the generalization of certain results. If for example, one came across certain results involving the construction of equilateral triangles or squares, one might then consider the possible extention of those results to the construction of similar isosceles triangles or similar rectangles and rhombi. Several examples of these will be given later on.

A useful global perspective

A hierarchical classification provides a useful global perspective which may lead to a more *cohesive* perspective on the underlying relationships between concepts, and therefore also to better *retention*. In addition, it is aesthetically pleasing and insightful to see how the various intersections between more general concepts produce the properties of more special concepts.

For example, since the rhombi are the intersection between the kites and parallelograms, it immediately follows from the diagonal properties of kites and parallelograms that the diagonals of a rhombus would perpendiculary bisect each other.

Similarly, since the rectangle are the intersection between the parallelograms and isosceles trapezia, it immediately follows that a rectangle must have opposite angles equal (parallelogram

property) as well as adjacent angles equal (isosceles trapezium property), from which we get the familiar property that all its angles are equal. In the same way, the rectangles "*inherit*" equal diagonals from the isosceles trapezia, as well as bisecting ones from the parallelograms.

In conclusion we therefore see that the reason why a hierarchical classification of quadrilaterals is generally preferred to a partition classification is that the former is much more useful and not because the latter is "*wrong*".

Questions and Problems 2

1. List **all** the properties of any quadrilateral, say a parallelogram. Is it possible to prove **all** the properties of a parallelogram?

2. State for each of the following conditions whether it is (verify with proofs or illustrate with counter-examples if necessary):
   (i) necessary and sufficient or
   (ii) necessary but not sufficient
   (iii) sufficient but not necessary

(Note : assume for this exercise that we're restricted to only convex and concave cases).

(a) A _____ condition that a kite is a rhombus is that it has one pair of opposite sides parallel.

(b) A _____ condition that a cyclic quadrilateral has at least one pair of opposite sides parallel, is that at least three of its sides must be equal.

(c) A _____ condition that a kite is cyclic is that it has at least one pair of opposite right angles.

(d) Equal diagonals is a _____ condition for a quadrilateral to be a rectangle.

(e) Equal diagonals is a _____ condition for a quadrilateral to be an isosceles trapezium.

(f) One pair of equal opposite angles is a _____ condition for a quadrilateral to be a kite.

(g) Three angles of a quadrilateral correspondingly equal to three angles of another quadrilateral, is a _____ condition for them to be similar.

(h) One pair of opposite sides parallel and one pair of opposite angles equal is a _____ condition for a quadrilateral to be parallelogram.

(i) One pair of opposite sides parallel and the other pair equal is a _____ condition for a quadrilateral to be a parallelogram.

(j) One pair of opposite sides equal and one pair of opposite angles equal is a

_____ condition for a quadrilateral to be a parallelogram.

(k) A point of half-turn symmetry is a _____ condition for a quadrilateral to be a parallelogram.

(l) Two axes of symmetry through each of the two pairs of opposite sides is a _____ condition for a quadrilateral to be a rectangle.

(m) Two axes of symmetry, one through a pair of opposite sides and another through a pair of opposite angles, is a _____ condition for a quadrilateral to be a square.

(n) Two pairs of opposite angles supplementary and one pair of opposite sides parallel is a _____ condition for a quadrilateral to be an isosceles trapezium.

(o) Perpendicular diagonals and one pair of opposite sides parallel is a _____ condition for a quadrilateral to be a rhombus.

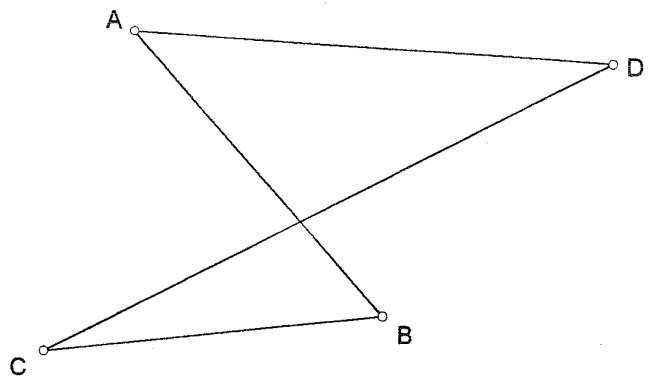
3. Critically evaluate each of the definitions below:
   (a) The squares are the set of quadrilaterals that are square.
   (b) The parallelograms are the set of quadrilaterals with all their sides parallel.
   (c) The rhombi are the set of quadrilaterals with all sides and angles equal.
   (d) The isosceles trapezia are the set of quadrilaterals with perpendicular diagonals.

4. Make a list of all the properties of an isosceles trapezium and then formulate:
   (i) an *incomplete* definition for an isosceles trapezium (one which has necessary but not sufficient properties).
   (ii) a correct, *uneconomical* definition for an isosceles trapezium.
   (iii) three correct, *economical* definitions for an isosceles trapezium.

5. Repeat the above exercise with rectangles, rhombi and squares.

6. Formulate (correct, economical) partition definitions for rectangles, rhombi, cyclic quads and bisecting quads.

7.(a) Carefully consider which angles are the *interior* angles of a crossed quadrilateral (see figure below). What is the sum of the interior angles of a crossed quadrilateral?



(b) Which are the *opposite* sides and angles of a crossed quadrilateral?

8. Try formulating alternative definitions for convex, concave and crossed quadrilaterals to those given the text.

9. Carefully consider whether we can find convex, concave or crossed examples of the following quadrilaterals (make drawings):
   (a) Bisecting quad
   (b) Perpendicular quad
   (c) Trapezium
   (d) Cyclic quad
   (e) Parallelogram
   (f) Right Kite
   (g) Rhombus
   (h) Rectangle

10. Repeat no. 2 with the additional consideration of crossed cases.

Note: In all the following exercises you have to consider convex, concave and crossed cases.

11. Are the kites the intersections between the following sets of quadrilaterals? (Check by construction and/or provide proofs. Give counter-examples where necessary).
   (a) the bisecting and perpendicular quads
   (b) the bisecting quads and skew kites
   (c) the angle quads and skew kites
   (d) the angle and the perpendicular quads
   (e) the skew and perpendicular quads
   (f) the bisecting and angle quads

12. Find a formula for the area of a convex perpendicular quad in terms of its diagonals. Is this formula also valid for the concave and crossed cases?

13. Prove that in a bisecting quad the diagonal that bisects the other also bisects its area.

14. Is a cyclic quadrilateral with equal diagonals always an isosceles trapezium? Investigate. If true, provide a proof. If not, provide a counter-example.

15. Is a trapezium with equal diagonals always an isosceles trapezium? Investigate. If true, provide a proof. If not, provide a counter-example.

16 What can you say about the sums of the two pairs of *alternate* sides of a kite $2n$-gon? (Alternate sides are those sides when moving along the perimeter of a polygon that are separated from each other by only one side; i.e. in other words, every second one).

17. Consider some ways of generalizing the concept parallelogram.

18. Is any quadrilateral ABCD an isosceles trapezium if it has opposite sides equal and the diagonals intersect in O so that AO/OC = DO/OB? Investigate. If true, provide a proof. If not, provide a counter-example.