9. The centroids of the four triangles determined by the vertices of a quadrilateral taken three at a time form a quadrilateral homothetic to the given quadrilateral. Find the center and the ratio of this homothecy.
10. From the midpoints A'', B'', C'' of the lines $B'C'$, $C'A'$, $A'B'$ joining the projections A', B', C' of a point P upon the sides of the triangle ABC, perpendiculars are dropped upon the sides BC, CA, AB of ABC. Show that these perpendiculars are concurrent. *Hint.* The two sets of perpendiculars are corresponding lines in the two homothetic figures $A'B'C'$ and $A''B''C''$.
11. If K, K' are two isotomic points on the side BC of the triangle ABC, and the line AK meets the side $B'C'$ of the complementary triangle in K'', show that the line $K'K''$ passes through the centroid G of ABC and that $K'G:GK'' = 2:1$. *Hint.* Consider the triangle AKK'.
12. Let L, L' and M, M' be two pairs of isotomic points on the two sides AC, AB of the triangle ABC, and L'', M'' the traces of the lines BL, CM on the sides $A'C'$, $A'B'$ of the complementary triangle $A'B'C'$ of ABC. Show that the triangles $AL'M'$, $A'L''M''$ are inversely homothetic. Find in the figure other pairs of homothetic triangles.
13. Construct a triangle given an angle, the sum of the two sides forming that angle, and the median to one of these sides $(A, b + c, m_c)$.

106. Theorem. *Twice the square of a median of a triangle is equal to the sum of the squares of the two including sides diminished by half the square of the third side.*

The proof was given in connection with the derivation of locus 10, § 11.

We have thus, in the usual notation:

$$2\,m_a^2 = b^2 + c^2 - \tfrac{1}{2}\,a^2, \quad 2\,m_b^2 = c^2 + a^2 - \tfrac{1}{2}\,b^2,$$
$$2\,m_c^2 = a^2 + b^2 - \tfrac{1}{2}\,c^2.$$

107. Theorem. *The sum of the squares of the medians of a triangle is equal to three-fourths the sum of the squares of the sides* (§ 106).

108. COROLLARY. *The sum of the squares of the distances of the centroid of a triangle from the vertices is equal to one-third the sum of the squares of the sides.*

109. Theorem. *If M is any point in the plane of the triangle ABC, centroid G, we have:*

$$MA^2 + MB^2 + MC^2 = GA^2 + GB^2 + GC^2 + 3\,MG^2.$$

If D is the midpoint of AG, we have, applying the formula for the median (§ 106) to the triangles MBC, MDA', MAG:

$$MB^2 + MC^2 = 2\,MA'^2 + \tfrac{1}{2}\,BC^2,$$
$$MD^2 + MA'^2 = 2\,MG^2 + \tfrac{1}{2}\,DA'^2,$$
$$MA^2 + MG^2 = 2\,MD^2 + \tfrac{1}{2}\,AG^2.$$

Now $DA' = AG$; hence, multiplying the second equality by 2 and adding, we obtain, after simplification:

$$MA^2 + MB^2 + MC^2 - 3\,MG^2 = \tfrac{1}{2}\,BC^2 + \tfrac{3}{2}\,AG^2.$$

Considering, in turn, the medians BB' and CC', we obtain two analogous relations, and adding the three, we have:

$$3(MA^2 + MB^2 + MC^2 - 3\,MG^2) = \tfrac{1}{2}(BC^2 + CA^2 + AB^2) + \tfrac{3}{2}(GA^2 + GB^2 + GC^2).$$

Observing that (§ 108):

$$BC^2 + CA^2 + AB^2 = 3(GA^2 + GB^2 + GC^2),$$

we arrive readily at the announced result.

110. COROLLARY. *If the point M* (§ 109) *coincides with the circumcenter O of ABC, we have:*

$$3\,R^2 = GA^2 + GB^2 + GC^2 + 3\,OG^2,$$

or:

$$OG^2 = R^2 - \tfrac{1}{9}(a^2 + b^2 + c^2).$$

EXERCISES

1. If two points are equidistant from the centroid of a triangle, show that the sums of the squares of their distances from the vertices of a triangle are equal, and conversely.
2. Construct a triangle, given a, m_a, $b + c$.
3. Find the locus of a point which moves so that the sum of the squares of its distances from two vertices of a given triangle is equal to the square of its distance from the third vertex of the same triangle.
4. A variable chord of a fixed circle subtends a right angle at a fixed point. Prove that the locus of the midpoint of the chord is a circle
5. If a, b, c are the sides, and S the area, of a given triangle, show that the area of the triangle formed by the feet of the perpendiculars from the centroid of the triangle upon its sides is equal to:

$$4\,S^3(a^2 + b^2 + c^2)/9\,a^2b^2c^2.$$

6. Construct a triangle, given the ratios $a:b$, $b:c$, and the sum of the medians.
7. If $A''B''C''$ is the anticomplementary triangle of the triangle ABC, and A' the midpoint of BC, show that:

$$A'B''^2 - A'C''^2 = 2(AB^2 - AC^2).$$