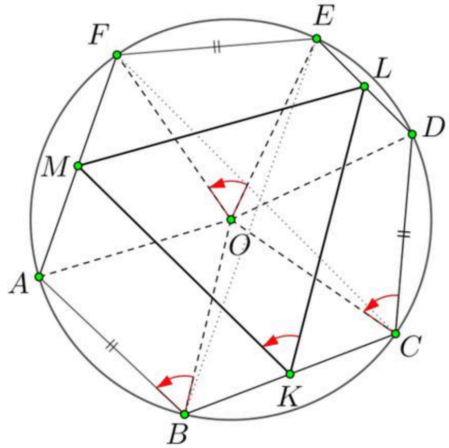
J701. In a circle of radius $R$, three chords of length $R$ are given. Their ends are joined with segments to obtain a hexagon inscribed in the circle. Show that the midpoints of the new chords are the vertices of an equilateral triangle.

*Proposed by Cristian Tudor Popescu, Bucharest, Romania*

*Solution 1 by Kousik Sett, India*

Let $O$ be the center of the given circle of radius $R$. If three chords are denoted by $AB$, $CD$, and $EF$ then by construction, we have triangles $OAB$, $OCD$, and $OEF$ are equilateral. Let $K$, $L$, and $M$ be the midpoints of $BC$, $DE$, and $FA$ respectively. We use vector rotation to prove this result.




We have

$$\overrightarrow{KL} = \frac{\overrightarrow{CD} + \overrightarrow{BE}}{2} = \frac{\overrightarrow{CD} + \overrightarrow{BO} + \overrightarrow{OE}}{2}.$$

We rotate each vector of the above equality by $60^\circ$ counterclockwise. We define $M'$ be a point such that triangle $KLM'$ is equilateral and after above rotation $\overrightarrow{KL}$ maps to $\overrightarrow{KM'}$. Thus each vector of the above equality transforms to

$$\overrightarrow{KM'} = \frac{\overrightarrow{CO} + \overrightarrow{BA} + \overrightarrow{OF}}{2} = \frac{\overrightarrow{BA} + \overrightarrow{CO} + \overrightarrow{OF}}{2} = \frac{\overrightarrow{BA} + \overrightarrow{CF}}{2} = \overrightarrow{KM},$$

which implies $M' \equiv M$. Therefore, triangle $KLM$ is equilateral and we are done!

*Solution 2 by Theo Koupelis, Clark College, Washington, USA*

Let $AB = CD = EF$ be the three chords of length $R$. Let $\angle BOC = 2\alpha$, $\angle DOE = 2\beta$, and $\angle FOA = 2\gamma$, and Let $K, L, M$ be the midpoints of the chords $BC, DE, FA$, respectively. Then $OK = R\cos\alpha$, $OL = R\cos\beta$, and $OM = R\cos\gamma$. Triangles $AOB, COD, EOF$ are equilateral, and thus $\angle AOB = \angle COD = \angle EOF = 60^\circ$. Therefore, $180^\circ + 2\alpha + 2\beta + 2\gamma = 360^\circ$, and thus $\alpha + \beta + \gamma = 90^\circ$. Using the law of cosines in triangles $KOL, KOM$ we get

$$KL^2 = OK^2 + OL^2 - 2 \cdot OK \cdot OL \cdot \cos(60^\circ + \alpha + \beta),$$
$$KM^2 = OK^2 + OM^2 - 2 \cdot OK \cdot OM \cdot \cos(60^\circ + \alpha + \gamma).$$

Thus, $KL = KM$ if and only if

$$\cos^2\beta - 2\cos\alpha\cos\beta\cos(60^\circ + \alpha + \beta) = \cos^2\gamma - 2\cos\alpha\cos\gamma\cos(60^\circ + \alpha + \gamma),$$

or

$$\begin{aligned} KL = KM &\iff \cos^2\beta - \cos^2\gamma = \cos\alpha\left[\cos(60^\circ + \alpha + 2\beta) - \cos(60^\circ + \alpha + 2\gamma)\right] \\ &\iff \frac{\cos(2\beta) - \cos(2\gamma)}{2} = -2\cos\alpha\sin(60^\circ + \alpha + \beta + \gamma)\sin(\beta - \gamma) \\ &\iff \cos(2\beta) - \cos(2\gamma) = -2\cos\alpha\sin(\beta - \gamma) = -2\sin(\beta + \gamma)\sin(\beta - \gamma), \end{aligned}$$

which is obvious. Similarly, we get $KL = LM$, and the triangle $KLM$ is equilateral.

**Note:** With appropriate signs for the angles $\alpha, \beta, \gamma$, the above holds true whether the hexagon $ABCDEF$ is convex or not.

*Solution 3 by Polyahedra, Polk State College, FL, USA*
This is well known as the asymmetric propeller theorem with many generalizations and various proofs (Euclidean, vector, and complex-number). It even appeared as Problem B-1 in the 1967 Putnam Competition. For its history, see M. Gardner, The asymmetric propeller, *The College Math. Journal*, January/1999, 18–22. For most recent generalizations, see Q. H. Tran, The asymmetric propeller with squares, and some extensions, *The Math. Gazette*, July/2024, 283–291.

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