Viviani's Theorem – A Geometrical Diversion

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Viviani's theorem, named after the Italian mathematician and geometer Vincenzo Viviani (1622 – 1703), is a simple yet delightfully intriguing geometry theorem with a number of interesting extensions. Viviani's theorem states that for *any* point inside an *equilateral* triangle the sum of the perpendiculars from that point to the sides of the triangle is equal to the length of the triangle's altitude.

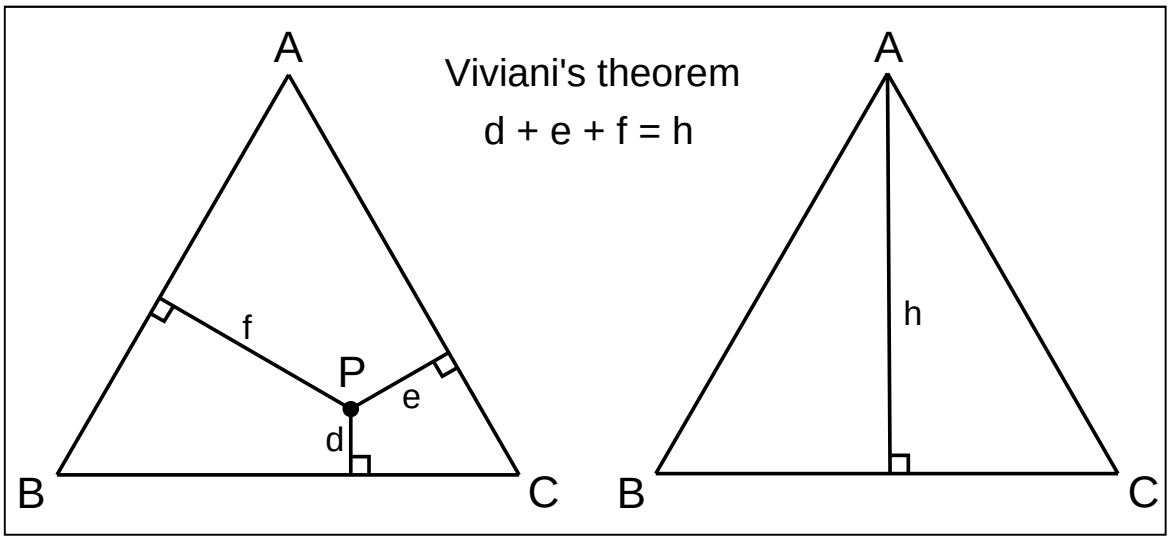

FIGURE 1: Viviani's theorem: $d + e + f = h$.

In Figure 1, irrespective of where point P is placed inside equilateral triangle ABC, the sum $d + e + f$ will be a constant value. This can be beautifully illustrated using Dynamic Geometry Software (DGS) where one can drag point P to different positions and see that the sum of the three perpendiculars remains constant. Furthermore, not only is this sum a constant value, but that constant value is the same as the perpendicular height (or altitude) of the triangle: $d + e + f = h$. Since the triangle is equilateral, the altitude from all three vertices is of course the same. As an interesting aside, it is worth noting that constructing the three perpendiculars from point P subdivides ΔABC into three cyclic quadrilaterals.

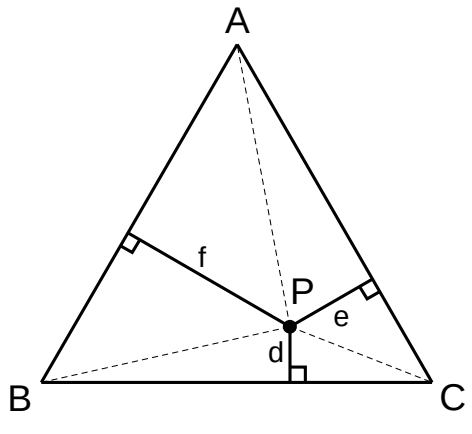

FIGURE 2: Proving Viviani's theorem.

Proving Viviani's theorem is a relatively simple matter. Firstly, join point P to each of the three vertices, thus forming lines PA, PB and PC. The original equilateral triangle ABC has now been subdivided into three smaller triangles: APB, BPC and APC. The sum of the areas of these three triangles is clearly the same as the area of the original equilateral triangle. This gives $\frac{1}{2}(BC)(d) + \frac{1}{2}(AC)(e) + \frac{1}{2}(AB)(f) = \frac{1}{2}(BC)(h)$ where h is the altitude of the original equilateral triangle. Since ΔABC is equilateral, $AB = BC = AC$, from which it follows that $d + e + f = h$.

There is another rather elegant way to prove Viviani's theorem which I first came across as a proof without words (Kawasaki, 2005). The proof shown in Figure 3 is a slight modification of Kawasaki's (2005) original proof without words.

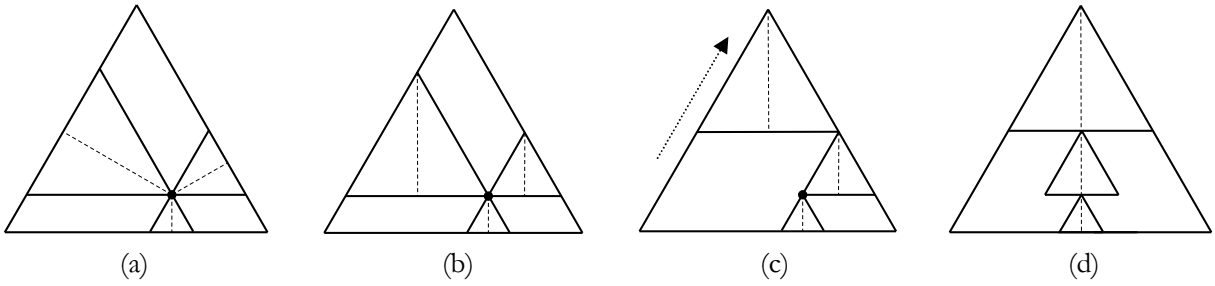

FIGURE 3: Viviani's theorem – a proof without words.

The proof proceeds as follows. Having chosen an arbitrary point P inside the equilateral triangle, draw in the perpendiculars from that point to the sides of the triangle. Now create three new equilateral triangles by drawing in three lines passing through the chosen point P such that each line is parallel to one of the three sides of the original equilateral triangle. The perpendiculars from the chosen point P will be the altitudes of these three new equilateral triangles (Figure 3a). Next rotate the triangles so that the respective altitudes are vertically oriented (Figure 3b). Slide the leftmost of the three triangles along the edge of the original equilateral triangle until their upper vertices coincide (Figure 3c). Finally slide the bottom two triangles horizontally to complete the proof that the sum of the three perpendiculars is equal to the altitude of the original equilateral triangle (Figure 3d).

The dynamic nature of this proof is incredibly powerful. When I workshop this particular proof I usually prepare an A4 sheet with a large equilateral triangle printed on it. Each learner is then given two copies of the equilateral triangle, one on white paper and the other on brightly coloured paper. Learners then line the two sheets up so that the two equilateral triangles overlap, and mark the point P using a pair of dividers (or something similar). This ensures that the chosen point P is in exactly the same place in both triangles. Learners then make a duplicate copy of their construction (Figure 3a), one on the white paper and the other on the brightly coloured sheet. The three smaller equilateral triangles formed are then cut out from the coloured sheet and placed over their matching positions on the white paper. The dynamic nature of the proof can now be enjoyed to maximum effect, since the three equilateral triangles can be rotated (Figure 3b), translated (Figure 3c), and finally slid horizontally (Figure 3d). The final denouement of the proof, when the three triangles line up vertically as in Figure 3d, never ceases to amaze and delight!

To recap, Viviani's theorem states that for any point inside an equilateral triangle the sum of the perpendiculars from that point to the sides of the triangle is not only invariant, but equal in length to the triangle's altitude. This equality still holds even if the chosen point lies *on* one of the three sides of the equilateral triangle (although in such cases one of the perpendiculars collapses to zero). But what if the chosen point P lies *outside* the original triangle? There is still an interesting relationship between the three perpendiculars from P and the altitude of the original triangle, but the relationship changes depending on the exact position of P. If we extend the three sides of the original triangle, then the space outside the triangle is effectively subdivided into six distinct regions (Figure 4).

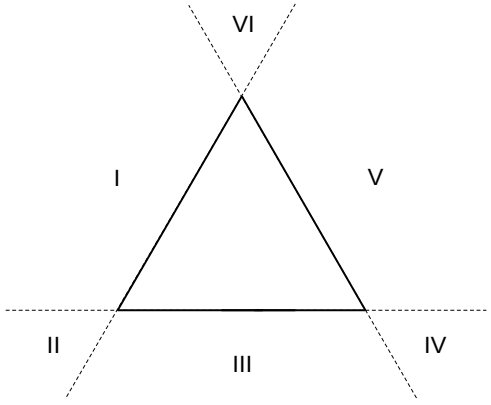

FIGURE 4: Extending the three sides.

Let's choose a point P in region V and construct perpendiculars from point P to each of the three sides as before. Since P lies outside the original equilateral triangle the perpendiculars are constructed to the *extensions* of two of the sides (Figure 5a). Next join P to each of the three vertices as before, thus forming lines PA, PB and PC (Figure 5b). We have now created three new triangles: APB, BPC and APC.

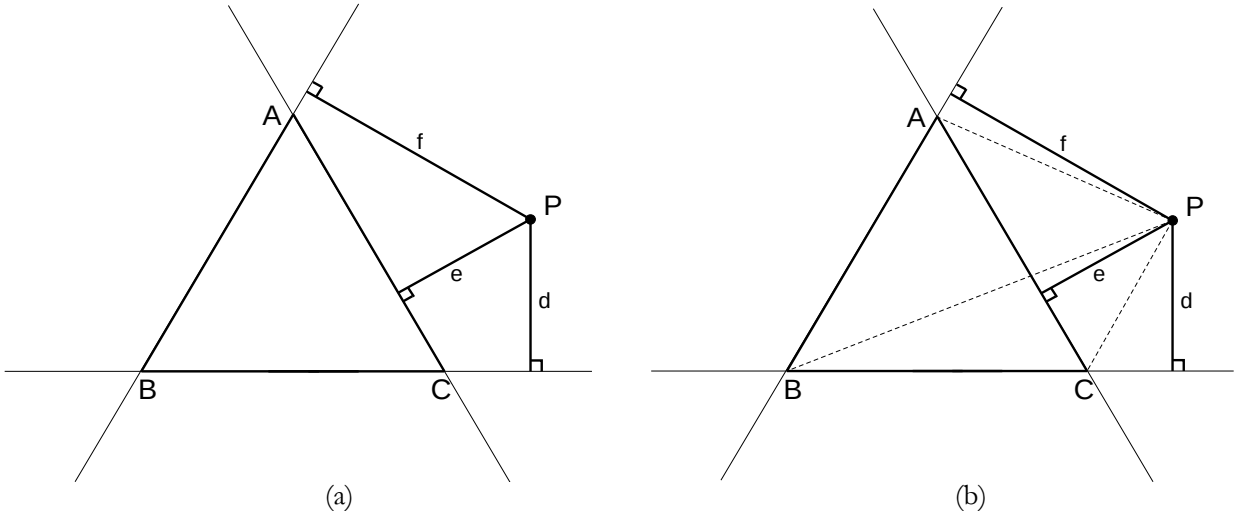

FIGURE 5: Choosing a point *outside* ΔABC.

In triangle APB, if we take AB as the base of the triangle, then f is the perpendicular height. The area of ΔAPB is thus $\frac{1}{2}(AB)(f)$. Similarly, the areas of ΔBPC and ΔAPC are $\frac{1}{2}(BC)(d)$ and $\frac{1}{2}(AC)(e)$ respectively. Comparing these three areas to that of the original equilateral triangle reveals the following relationship: $Area\ \Delta APB + Area\ \Delta BPC - Area\ \Delta APC = Area\ \Delta ABC$. Using the above expressions for the areas of the various triangles we thus have $\frac{1}{2}(AB)(f) + \frac{1}{2}(BC)(d) - \frac{1}{2}(AC)(e) = \frac{1}{2}(BC)(h)$ where h is the altitude of the original equilateral triangle. Since ΔABC is equilateral, $AB = BC = AC$, from which it follows that $f + d - e = h$. Similar but different expressions can be derived for each of the other five regions. Alternatively, instead of considering each region separately, one can introduce the concept of *directed distances*, and thus view heights that fall completely outside the triangle as being negative (thus resulting in negative areas), thereby ensuring that the sum $d + e + f$ remains constant irrespective of whether P lies inside or outside the original equilateral triangle (De Villiers, 2003, p. 149).

Let's leave Viviani's theorem for a moment and make a slight diversion. Take a regular hexagon and place a point P anywhere inside the hexagon. Next join point P to each of the six vertices of the hexagon (Figure 6a) thereby splitting the original hexagon into six triangles.

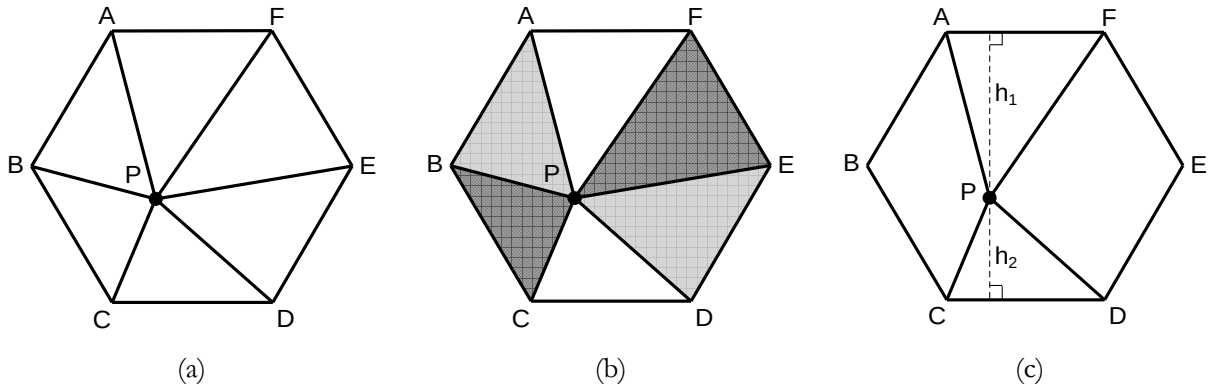

FIGURE 6: Slicing up a regular hexagon.

An interesting result of this construction is that if each triangle is paired with the triangle opposite it (Figure 6b) then each pair of triangles will have the same constant area, irrespective of the initial position of point P. In an analogous situation, imagine a cake in the shape of a regular hexagon. Place a knife at a random point P on the cake and make cuts from P to each corner. If three people now each take a slice of cake along with the slice diametrically opposite it, they will all end up with the same amount of cake irrespective of where the knife was initially placed. The reason for this can readily be seen from Figure 6c. The pair of triangles highlighted in Figure 6c is ΔAPF and ΔCPD. The altitudes from point P have been drawn in for each of the two triangles. The area of ΔAPF plus the area of ΔCPD can be expressed as $\frac{1}{2}(AF)(h_1) + \frac{1}{2}(CD)(h_2)$. Similar area expressions can be obtained for the other two pairs of triangles. Now, since the original hexagon is *regular*, all six sides are equal in length, opposite sides are parallel, and the perpendicular distance between opposite sides is constant. From these properties it should be clear why each pair of opposite triangles will have the same area.

Returning to our original slicing of the hexagon (Figure 6a), another interesting property of this slicing is that the six triangles formed can be grouped into two groups of three so that each group has the same total area (Figure 7). If we use the cake analogy again, if two people each take alternate slices then they will each end up with the same amount of cake - a somewhat surprising result.

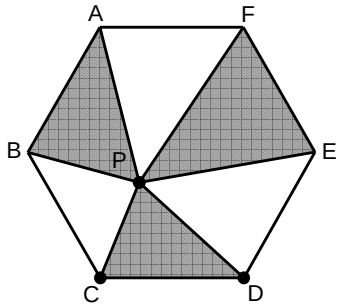

FIGURE 7: Two groups of three.

At this point you should be wondering what all this has to do with Viviani's theorem. The connection is rather beautiful, and is based on the fact that if one takes a regular hexagon and extends each of the six sides in both directions, each extension being the same length as the sides of the original hexagon, then one creates two identical overlapping equilateral triangles which bound the sides of the hexagon (Figure 8). The six perpendiculars from P have been drawn in and are indicated as h_1 to h_6.

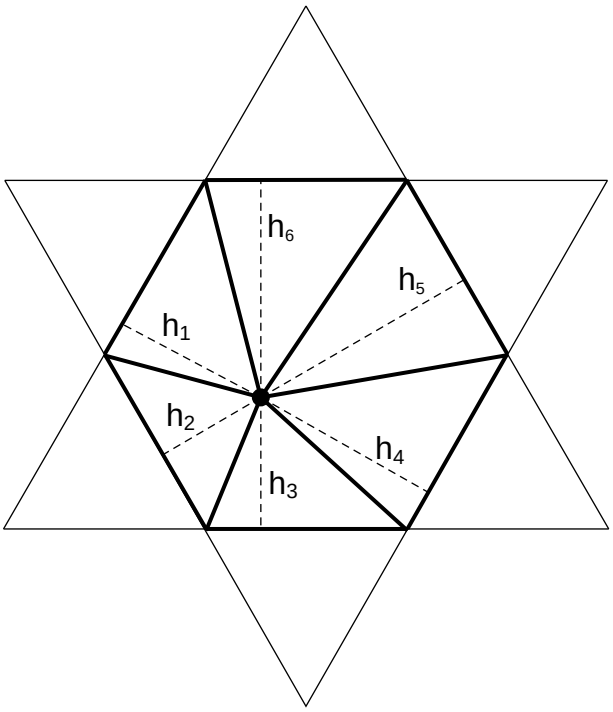

FIGURE 8: Viviani's theorem strikes again.

Since the hexagon is regular, let us for the sake of simplicity refer to the length of each side of the nested hexagon as s. The sum of the areas of the three shaded triangles highlighted in Figure 7 is thus $\frac{1}{2}(s)(h_1) + \frac{1}{2}(s)(h_3) + \frac{1}{2}(s)(h_5)$ which in factorised form is $\frac{1}{2}(s)(h_1 + h_3 + h_5)$, and from Viviani's theorem we know that this equals $\frac{1}{2}(s)(H)$ where H is the height of the large equilateral triangle. Using a similar argument, the sum of the areas of the three un-shaded triangles is $\frac{1}{2}(s)(h_2) + \frac{1}{2}(s)(h_4) + \frac{1}{2}(s)(h_6)$ which similarly is also equal to $\frac{1}{2}(s)(H)$. And that rather beautifully completes the explanation.

A final observation with regard to Figure 8 is that the six triangles formed around the perimeter of the original hexagon are all identical equilateral triangles whose sides are the same length as the sides of the original regular hexagon. It thus follows that (i) the sum of the areas of these six triangles equals the area of the enclosed hexagon, (ii) the area of the hexagon is two thirds of the area of one of the large equilateral triangles, and (iii) the area of the three shaded triangles not only equals the area of the three un-shaded triangles (Figure 7) but also equals the area of three of the smaller identical equilateral triangles around the perimeter of the enclosed hexagon (Figure 8). It is left to the reader to confirm the veracity of these last three observations.

Concluding comments

Viviani's theorem, although simple in statement, holds a wealth of potential in terms of mathematical investigation as well as proof/justification (see for example Copes & Kahan, 2006). In this article I have focused on the equilateral triangle, but the same basic principle (i.e. the invariance of the sum of the perpendiculars from an arbitrarily positioned interior point) can be extended to any equilateral or equiangular polygon (see for example De Villiers, 2005). In the case of regular polygons, i.e. polygons that are both equilateral *and* equiangular, the sum of the perpendiculars from an arbitrary interior point is not only invariant, but for a regular *n*-sided polygon the sum will equal *n* times the apothem (i.e. the perpendicular distance from the *centre* of the polygon to the midpoint of a side). It is left for the reader to discover why, in the case of a parallelogram (which need not be either equiangular or equilateral), the sum of the perpendiculars to the sides (or extensions of the sides if necessary) from an interior point is also independent of the position of the interior point.

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