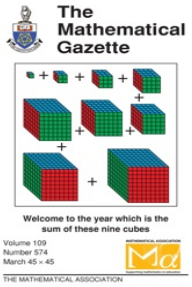

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Problem Corner

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Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 December 2007.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

91.E (Stan Dolan)

One of Fermat's most fully explained applications of his method of infinite descent was his proof that the area of a right-angled triangle with rational side-lengths cannot be a rational square.

Show that this result does *not* extend to general triangles with rational side-lengths. Specifically, find an infinite family of dissimilar triangles with rational side-lengths all of whose areas are rational squares.

91.F ((a) Nick MacKinnon, (b) Prithwijit De)

(a) Which positive integers a, b, c are solutions of the equations $(a^4 + b^4 + c^4)^2 = 2(a^8 + b^8 + c^8)$?

(b) Let N be an odd positive integer. Show that the equation $x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 + x_6^4 + x_7^4 + x_8^4 = 8N + 1$ has no integer solutions.

91.G ((a) Victor Nicola, (b) Tony Robin)

(a) The Lucas sequence (L_n) is defined by $L_1 = 1$, $L_2 = 3$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geqslant 3$. Give a quick proof that L_{2n} is not a perfect square.

(b) In the Maclaurin expansion of $\ln(1 - 2x + x^3)$, show that the coefficient of x^{2^n} is always an even integer.

91.H (Ovidiu Furdui)

Let $E_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$ and $e_n = \left(1 + \frac{1}{n}\right)^n$.

Calculate $\lim_{n \to \infty} n\left(e_n^e - e^{E_n}\right)$.

Solutions and comments on **90.I, 90.J, 90.K, 90.L** (November 2006).

90.I (Michael de Villiers and Michael Fox)

The diagonals of a cyclic quadrilateral $ABCD$ meet at P. The feet of the perpendiculars from P to AB, BC, CD, DA are E, F, G, H. The lines EP, FP, GP, HP meet the opposite sides of $ABCD$ at K, L, M, N. Prove that E, F, G, H, K, L, M, N lie on a conic.

This attractive geometric result produced an interesting range of Euclidean and projective solutions and inspired John Rigby to produce the definitive analysis and beautiful generalisation which follows.

Definition: If two pairs of lines a, b and m, e are all concurrent, and if $\angle am = -\angle be$ (as in Figure 1 for example), then the two pairs are *equi-inclined* to each other. The condition can equally well be written $\angle ae = -\angle bm$, so this is a relation between unordered pairs: e, m is the same pair as m, e.

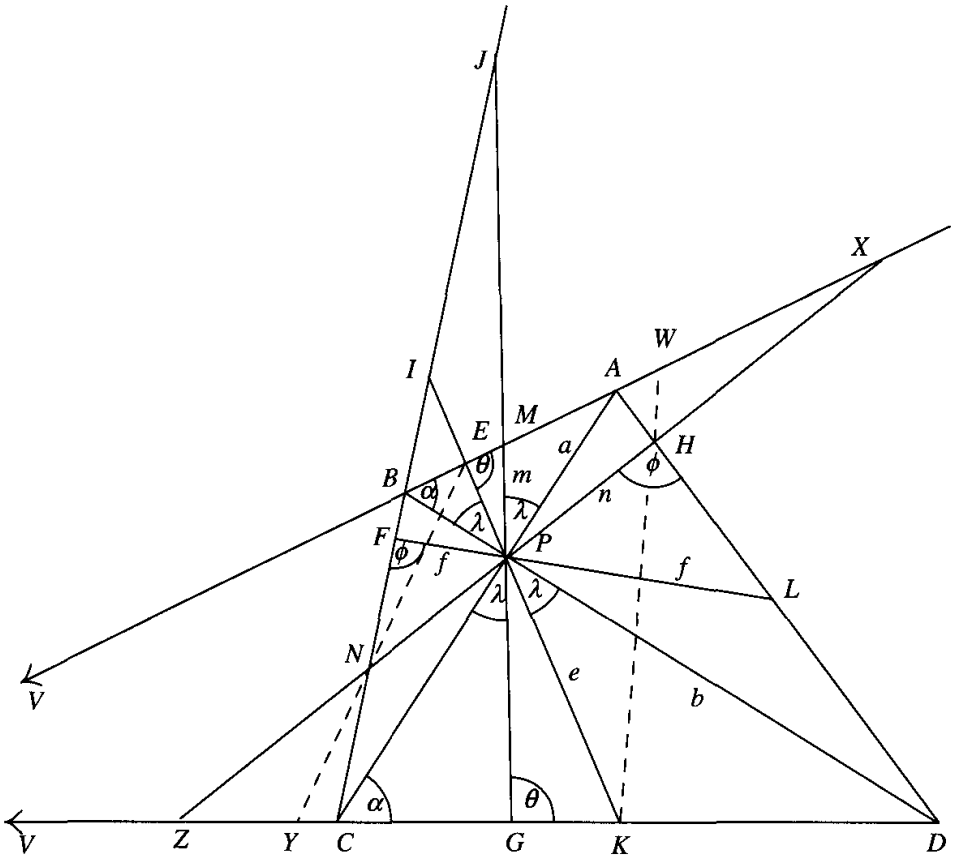

FIGURE 1

In Figure 1, the diagonals $AC = a$ and $BD = b$ of the quadrilateral $ABCD$ meet at P, and four lines through P meet the side lines of the quadrilateral as shown. Denote the lines MPG, NPH, etc. by m, n, etc. (All lines in the problem are infinite in extent.) If $ABCD$ is cyclic, the angles $\angle PBA$ and $\angle DCP$, marked α in the figure, are equal; if also the two angles marked θ are equal (but not necessarily equal to 90° as in the statement of the problem), then the angles marked λ are equal, so the pairs of lines a, b and m, e are equi-inclined to each other. Similarly if $ABCD$ is cyclic and the angles marked ϕ are equal, then the pairs a, b and n, f are equi-inclined.

We shall show that if m, e and n, f are equi-inclined to the pair of diagonals a, b, then the eight points E, F, G, H, K, L, M, N lie on a conic for *any* quadrilateral $ABCD$ (not necessarily cyclic). But there is one further

generalisation (Theorem 1), which will turn the result into a theorem of projective geometry with a projective proof. For this we need more definitions. A *projectivity* from a line x to a line y is a one-one mapping of the set of points on x onto the set of points on y that preserves cross-ratios. An *involution* of points on a line is a projectivity from a line to itself of order 2, so if an involution maps X to Y then it maps Y to X, and X and Y are called a *pair of mates* in the involution. There are corresponding definitions for involutions of lines through a point. We shall quote various results about involutions, including the following lemmas.

Lemma 1: Let X, Y, P, Q be collinear points such that $X \neq P$, $X \neq Q$, $Y \neq P$, $Y \neq Q$. Then there exists a unique involution having (X, Y) and (P, Q) as pairs of mates.

Lemma 2: If the concurrent pairs of lines m, e and n, f are equally inclined to the pair a, b, then they are equally inclined to each other, and the pairs are three pairs of mates in an involution.

Theorem 1: Let the diagonals $AC = a$ and $BD = b$ of the quadrilateral $ABCD$ meet at P, and let (a, b), (e, m), (f, n) be three pairs of mates in an involution of lines through P. Let e, m meet AB at E, M, and meet CD at K, G, and let f, n meet BC at F, N and meet DA at L, H, as in Figure 1. Then E, F, G, H, K, L, M, N lie on a conic.

Proof: The three pairs of mates in the given involution at lines through P meet BC in three pairs of points (C, B), (I, J), (F, N) in an involution, ω_1 say, on the line BC. By Desargues' involution theorem, all the conics through M, E, G, K, including the line-pairs (EM, GK) and (EK, GM) meet BC in pairs of mates in an involution, ω_2 say; two of these pairs are (B, C) and (I, J), and hence $\omega_2 = \omega_1$ by the uniqueness in Lemma 1. So (F, N), a pair of mates in ω_1, is now a pair of mates in ω_2 also. And by the converse of Desargues' involution theorem any pair of mates in ω_2, in particular (F, N), lie on a conic, Σ say, through M, E, G, K. It remains only to prove that Σ passes through L and H also; this will follow from Theorem 2, a nice little result in its own right.

Theorem 2: In Figure 1, let EK, MG, NH be any lines through the meet P of the diagonals of $ABCD$, meeting the sides as shown; then M, E, N, G, K, H lie on a conic.

Proof: Let AB and CD meet at V, and define the remaining points as shown. Then, using the notation $(VZ, CY) =_N (VX, BE)$ to signify that the cross-ratios (VZ, CY) and (VX, BE) are equal by projection from N, we have

$$(VZ, CY) =_N (VX, BE) =_P (VZ, DK) =_H (VX, AW),$$

and hence ZX, CA, YW are concurrent, so YW passes through P. Hence the three meets of opposite sides of the hexagon $ENHKGM$, namely Y, P, W,

are collinear, so the vertices of the hexagon lie on a conic by the converse of Pascal's theorem.

Now, in Theorem 1, Σ is the conic through *MEGKN*, which passes through *H* by Theorem 2; similarly , which passes through *L*, also by Theorem 2.

When we have proved Theorem 1, its converse follows easily; it states that:

'If the diagonals of *ABCD* meet at *P*, if four lines through *P* meet the sides of *ABCD* at eight points *M*, *E*, ... (using the notation of Figure 1), and if these eight points lie on a conic, then (*a*, *b*), (*e*, *m*), (*f*, *n*) are pairs of mates in an involution.'

This converse shows that Theorem 1 provides the best possible generalisation of the original result, because it gives us the most general conditions under which the four lines *e*, *m*, *f*, *n* through *P* meet the sides of *ABCD* at eight points lying on a conic. Finally, Theorem 3 shows that four lines through a point *P* cannot meet the sides of *ABCD* at eight points on a conic *unless P is the meet of the diagonals.*

Theorem 3: Let *ABCD* be a quadrilateral, suppose *PEK*, *PFL*, *PGM*, *PHN* are four lines through a point *P* meeting *AB* at *E*, *M*, meeting *BC* at *F*, *N*, meeting *CD* at *G*, *K*, and meeting *DA* at *H*, *L*, and suppose that the eight points of meeting lie on a conic. Then *P* is the meet of the diagonals *AC* and *BD*.

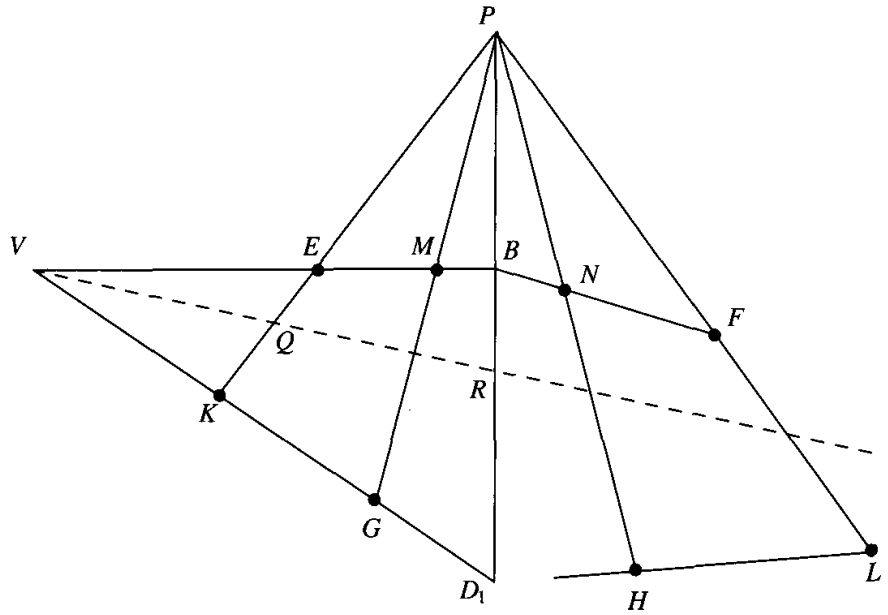

FIGURE 2

Proof: The redrawn version of the figure (Figure 2) helps to show the idea behind the proof: if we think of *P* as being at infinity in the direction perpendicular to its polar, the figure becomes symmetrical about the polar. The conic has been omitted from the figure for the sake of clarity.

The points $P = EK \cap MG$ and $V = EM \cap KG$ are two vertices of the diagonal triangle of quadrangle $MEGK$, which is inscribed in the conic; this triangle is self-polar, so V lies on the polar of P. Let PEK and PB meet the polar of P at Q and R, and let PB meet KG at D_1. Then $(PR, BD_1) =_V (PQ, EK) = -1$. Similarly if BP meets LH at D_2, $(PR, BD_2) = -1$. Hence $D_1 = D_2 = D$, so BD passes through P. Similarly AC passes through P.

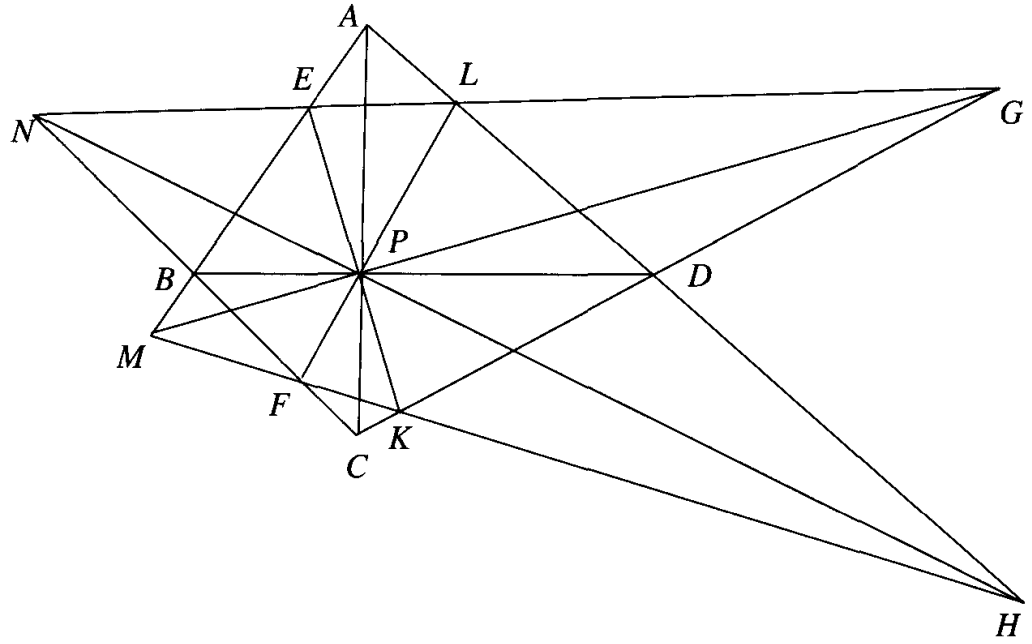

FIGURE 3

The original problem is a special case of Theorem 1. There are various other special cases; here is a nice one, in which the involution is chosen to be the involution of pairs of perpendicular lines through P. Let $ABCD$ be a quadrilateral with perpendicular diagonals meeting at P. Let the perpendicular lines EPK and MPG meet AB and CD as shown in Figure 3, and let MK meet BC at F. Let the perpendicular lines FPL and NPH meet BC and DA as shown. Then the conic through $EFGHKLMN$ consists of two lines since we have chosen F so that F, K, M are collinear: we have M, F, K, H collinear and N, E, L, G collinear. The figure could equally well be built up starting with the quadrilateral $GEMK$ or $NFLH$, both of which have perpendicular diagonals. Or, start with the three pairs of perpendicular lines through P, and choose A, M, H arbitrarily on three of the lines; then the remaining points and lines in the figure fall into place automatically.

Correct solutions were received from G. Howlett, J. F. Rigby, G. N. Thwaites and the proposers, M. de Villiers and M. D. Fox.