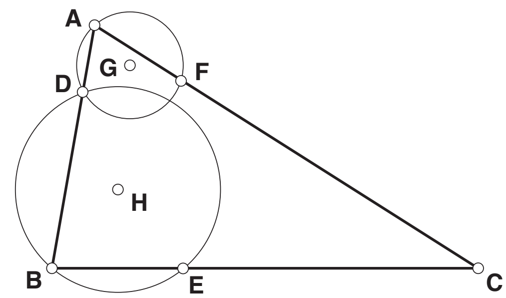
Miquel

In this investigation, you will explore a construction based on arbitrary points on the sides of an arbitrary triangle and some circles related to these points. The result you will find was apparently first discovered by a French mathematician named Auguste Miquel in 1838.



CONJECTURE

- Open the sketch **Miquel.gsp.** Drag different points to familiarize yourself with the sketch.

1. Explain the locations of points *G* and *H*.

- Press the button to show the circle through the points *F, E,* and *C* and its center *I*.

2. What do you notice about the three circles?

- Drag any of the points *D, E,* and *F* to check or change your observation.
- Also change the shape of ΔABC by dragging any vertex to check or change your observation.
- Press the button to show ΔGHI.

To find the ratio between two segment lengths, select both segments, then choose **Ratio** from the Measure menu.

3. Drag point *A, B,* or *C*. What do you notice about the shape of ΔGHI? (Take measurements, if necessary, to confirm your guess.)

- Drag point *D, E,* or *F* to check or change your observation.
- Also change the shape of ΔABC by dragging any vertex to check or change your observation in Question 3.

CHALLENGE Can you prove either of your conjectures from Question 2 and Question 3? (*Hint:* Use the property that a cyclic quadrilateral—a convex quadrilateral inscribed in a circle—has opposite angles that are supplementary.)

PROVING

You should have found the surprising results that the three circles are always concurrent at a point and the centers *G*, *H*, and *I* form a triangle similar to ΔABC. We can state these two separate conjectures in the following way.

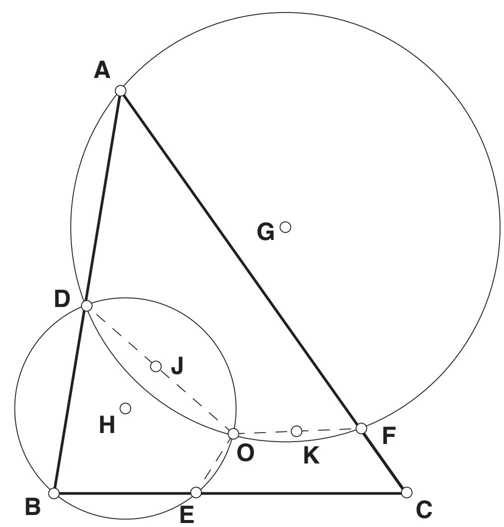
If three points *D*, *E*, and *F* are constructed on the sides of any triangle *ABC*, with *D* on $\overline{AB}$, *E* on $\overline{BC}$, and *F* on $\overline{CA}$, then

- The circumcircles of triangles *ADF*, *BDE*, and *CEF* are concurrent.
- The circumcircles of triangles *ADF*, *BDE*, and *CEF* form a triangle similar to ΔABC.

The hints that follow will help you prove these observations. Read and work through them carefully.

PROVING CIRCUMCIRCLES CONCURRENT

- Press the button that hides circle *FEC*.
- Press the button that shows segments *OD*, *OE*, and *OF* as well as points *J* and *K*.
- Press the button that hides ΔGHI.



We will first prove that quadrilateral *OECF* is cyclic, which implies that the three circles *ADF*, *BDE*, and *CEF* are concurrent at *O*.

4. Express the measure of angle *DOF* in terms of the measure of angle *A*. Give a reason for your equation.

5. Express the measure of angle *DOE* in terms of the measure of angle *B*. Give a reason for your equation.

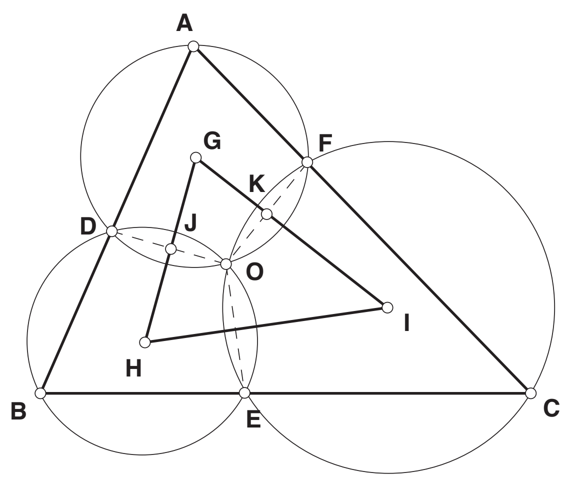
6. Using Questions 4 and 5, determine the measure of angle *EOF*. Give a reason.

7. Use Question 6 and your knowledge of the sum of the angle measures of a triangle to express the measure of angle *EOF* in terms of the measure of angle *C*.

8. From Question 7 and your knowledge of the sum of the angle measures of a triangle, what can you now conclude about quadrilateral *OECF*? Why?

PROVING TRIANGLE *GHI* SIMILAR TO TRIANGLE *ABC*

Now you will prove your second conjecture. Use the buttons in your sketch to show the quadrilaterals as you need them in this next section. The quadrilaterals blink, but then eventually remain hidden so as not to clutter your sketch.



Show Δ *GHI*.

9. What type of quadrilateral is *GDHO*? Why?

10. From Question 9, what can you now say about ∠*GJO*? Why?

11. What type of quadrilateral is *GOIF*? Why?

12. From Question 11, what can you now say about ∠*GKO*? Why?

13. From Questions 10 and 12, what can you conclude about quadrilateral *GJOK*? Why?

14. From Question 4 in the previous proof, and Question 13 on the previous page, what can you now conclude regarding $\angle JGK$? Why?

15. Repeat Questions 9–14 for either of the other angles of ΔGHI.

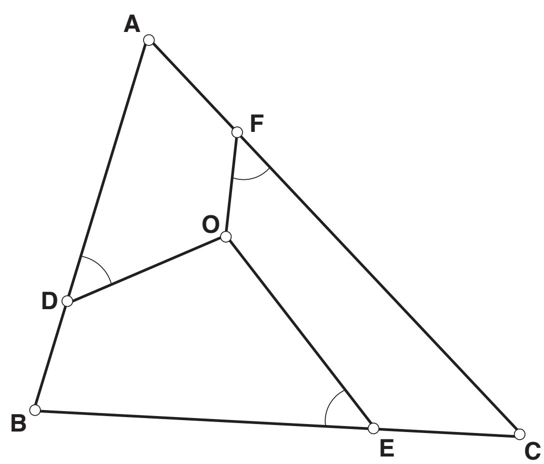
Present Your Proofs

Look over the steps for both proofs above. Now write a proof of one or both conjectures in your own words. You may include a demonstration sketch to support and explain each proof.

Investigate Further

1. Investigate what happens if point D lies on line AB but not necessarily between A and B. Are the results still valid if one or more of the points D, E, and F fall on extensions of the sides of ΔABC?

2. Start with an arbitrary point O in a triangle ABC and construct lines to make equal angles with the sides as shown below. (Use the sketch **Miquel 2.gsp.**) What conjectures can you make and prove?



MIQUEL (PAGE 122)

This worksheet focuses on further developing students' skills in writing proofs, rather than on the meaning of proof.

Prerequisites: Knowledge of the properties of cyclic quads, the AA condition of similarity, and the fact that the diagonals of a kite are perpendicular to each other.

Sketch: Miquel.gsp. Additional sketch is **Miguel 2.gsp.**

CONJECTURE

1. *G* and *H* are the centers of circles drawn, respectively, through points *A, D,* and *F* and points *D, B,* and *E*.
2. The three circles are always concurrent at a point.
3. The centers *G, H,* and *I* form a triangle similar to triangle *ABC*.

CHALLENGE This provides students with the opportunity to attempt their own proofs.

PROVING CIRCUMCIRCLES CONCURRENT

4. $m\angle DOF = 180° - m\angle A$, since *ADOF* is cyclic.
5. $m\angle DOE = 180° - m\angle B$, since *BEOD* is cyclic.
6. $m\angle EOF = 360° - (180° - m\angle A + 180° - m\angle B) = m\angle A + m\angle B$ (the sum of the measures of angles around *O* is 360°).
7. $m\angle EOF = 180° - m\angle C$, since $m\angle A + m\angle B + m\angle C = 180°$.
8. Quadrilateral *OECF* is cyclic (opposite angles are supplementary).

PROVING TRIANGLE *GHI* SIMILAR TO TRIANGLE *ABC*

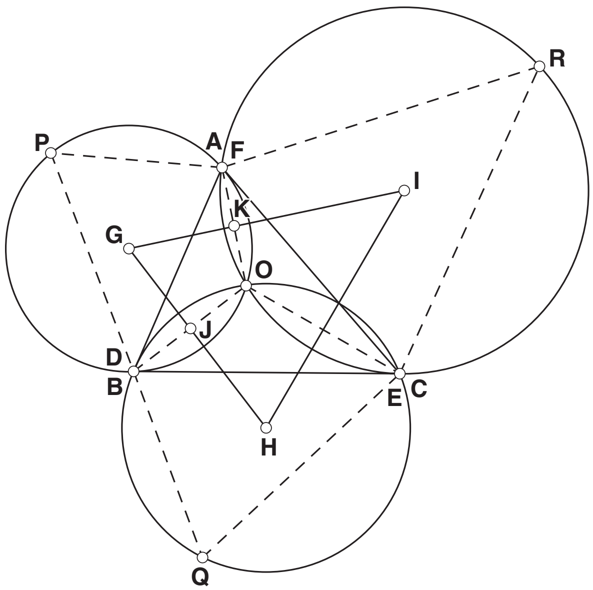
9. *GDHO* is a kite, since $\overline{GD}$ and $\overline{GO}$ are radii of circle *G*, and $\overline{HD}$ and $\overline{HO}$ are radii of circle *H*.
10. $m\angle GJO$ is 90°, since the diagonals of a kite are perpendicular to each other.
11. *GOIF* is a kite, since $\overline{GO}$ and $\overline{GF}$ are radii of circle *G* and $\overline{IO}$ and $\overline{IF}$ are radii of circle *I*.
12. $m\angle GKO$ is 90°, since the diagonals of a kite are perpendicular to each other.
13. Quadrilateral *GJOK* is cyclic, since opposite angles *GJO* and *GKO* are supplementary.
14. $m\angle DOF = 180° - m\angle A$ from the first proof. But angle *JGK* and angle *JOK* are opposite angles in cyclic quad *GJOK*. Therefore, $m\angle JGK = 180° - (180° - m\angle A) = m\angle A$.
15. Follows in the same way.

Investigate Further

1. The result still holds even if the constructed points lie on the extensions of the sides. Note, however, that the above proof is not general enough to cover the cases in which the points lie on the extensions of the sides or the point of concurrency lies outside the triangle, although it can easily be adapted. For a completely general proof covering all cases, we need to use the idea of directed line segments (for example, see Johnson 1929, 133).
2. Essentially, this is only the converse of Miquel's theorem, as quadrilaterals *ADOF, BEOD,* and *CFOE* are all cyclic (exterior angles are equal to opposite interior angles). Furthermore, exactly as before, their circumcircles intersect at *O* and the three centers of these circles form a triangle similar to triangle *ABC*.

Connecting to the Napoleon Generalization

In the Teacher Notes for the Napoleon activity, the following interesting generalization of Napoleon's theorem is mentioned: "If triangles *DBA, BEC,* and *ACF* are erected on the sides of any triangle *ABC* so that $m\angle D + m\angle E + m\angle F = 180°$, their circumcircles meet in a common point, and their circumcenters *G, H,* and *I* form a triangle, then $m\angle G = m\angle D$, $m\angle H = m\angle E$, and $m\angle I = m\angle F$."



Perhaps surprisingly, this generalization can be viewed as a special kind of limiting case of Miquel's theorem, as follows: In the **Miquel.gsp** sketch, drag *D* to (almost) coincide with *B,* drag *E* to (almost) coincide with *C,* and drag *F* to (almost) coincide with *A*. (Note that technically speaking, none of the circles is uniquely defined if all of these three points coincide exactly with the vertices of triangle *ABC*.)

From Miquel's theorem, we have triangle *GHI* similar to triangle *ABC,* and the circles *G, H,* and *I* concurrent at *O.* From the properties of cyclic quadrilaterals, it now follows that any angle *APB* would be equal to angle *G,* any angle *BQC* would be equal to angle *H,* and any angle *CRA* would be equal to angle *I*. But this configuration is equivalent to the above generalization of Napoleon's theorem.

From this formulation, the following converse of the above generalization of Napoleon's theorem is now also apparent: If for any arbitrary point *O* and triangle *ABC,* three circumcircles *AOB, BOC,* and *COA* are constructed, and triangles *DBA, BEC,* and *ACF* are erected so that *D, E,* and *F* are arbitrary points respectively in the arcs *AB, BC,* and *CA,* then the circumcenters *G, H,* and *I* form a triangle with $m\angle G = m\angle D$, $m\angle H = m\angle E$, and $m\angle I = m\angle F$ (and obviously $m\angle D + m\angle E + m\angle F = 180°$).