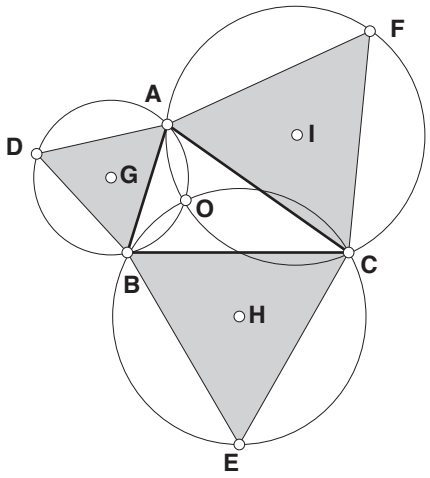
Napoleon



If you construct equilateral triangles on the sides of any triangle, you get some interesting results. You may have already discovered some of these if you worked through the previous activity The Fermat-Torricelli Point. One result this activity requires is that the three circumcircles of the equilateral triangles are always concurrent at a special point, called the Fermat-Torricelli point. This point is labeled *O* in the diagram.

In this activity, you'll discover another result related to this construction, attributed to Napoleon Bonaparte, the famous French emperor and general. Napoleon greatly enjoyed geometry and apparently discovered and proved this next conjecture.

CONJECTURE

- Open the sketch **Napoleon.gsp.** Drag different points in your sketch to become familiar with the construction.
- Use the **Polygon** tool to construct the triangle formed by centers *G, H,* and *I.*

1. Drag any vertex of ΔABC. What do you notice about ΔGHI? Take measurements if necessary and drag some more to confirm your conjecture.

2. Check your conjecture from Question 1 for the following special cases:

• Triangle *ABC* is obtuse.

• Points *A, B,* and *C* lie on the same line.

• The equilateral triangles lie inwardly and overlap.

Report your observations.

CHALLENGE Try to prove your conjecture from Question 1. (*Hints:* (1) Construct $\overline{AO}$, $\overline{BO}$, and $\overline{CO}$ and consider their relationship to the sides of ΔGHI. (2) Use the results proved in the activity The Fermat-Torricelli Point.) If you get stuck, go on to the more detailed hints that follow.

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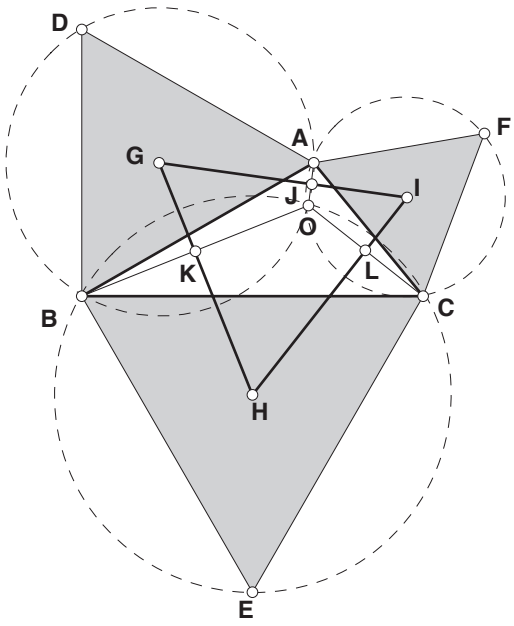
PROVING

In the preceding section, you should have found this surprising result:

If you construct equilateral triangles on each side of a triangle and then connect their centers, you form another equilateral triangle.



Below are some hints for planning a possible proof. Read and work through them carefully.

In previous work, we have already proved that the three circumcircles meet at the Fermat-Torricelli point *O*. We will now use the properties of cyclic quadrilaterals to show that each angle of ΔGHI measures 60°.

- Press the button that shows segments *AO, BO,* and *CO* and their intersections with the sides of ΔGHI.

Press the buttons in your sketch as necessary to see the quadrilaterals in the next few questions more clearly. The buttons eventually hide the quadrilaterals to keep your sketch from getting cluttered.

3. What type of quadrilateral is *ADBO*? Why?

4. From Question 3, what can you conclude about the measure of angle *AOB*? Why?

5. What type of quadrilateral is *GBHO*? Why?

6. From Question 5, what can you now conclude about the measure of angle *GKO*? Why?

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7. What type of quadrilateral is *GOIA*? Why?

8. From Question 7, what can you now conclude about the measure of angle *GJO*? Why?

9. What can you now conclude about angle *KGJ* in quadrilateral *GJOK*? Why?

10. Repeat Questions 3–9 for either of the other two angles of $\varnothing GHI$.

Present Your Proof

Look over Questions 3–10. Now write a proof of your original conjecture in your own words. You may include a demonstration sketch to support and explain your proof.

Further Exploration

Investigate what happens to ΔGHI if different arrangements of similar triangles are placed on the sides of ΔABC.

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NAPOLEON (PAGE 119)

This activity follows on The Fermat-Torricelli Point activity and further explores the properties of the same configuration. However, it is possible to do this activity independently of The Fermat-Torricelli Point activity.

Prerequisites: Knowledge of properties of a cyclic quadrilateral. Specifically, students should know that the opposite angles in a cyclic quadrilateral are supplementary, and they should also know the converse. Students should also be familiar with the definition of a kite and the property that its diagonals are perpendicular.

Sketch: Napoleon.gsp. Additional sketch is **Napolean 2.gsp.**

CONJECTURE

1. Triangle *GHI* is equilateral.
2. It remains equilateral.

CHALLENGE This provides students with an opportunity to attempt their own proofs.

PROVING

3. Quadrilateral *ADBO* is cyclic, because a circle passes through all four of its vertices.
4. Since *DBOA* is cyclic, angle *AOB* and angle *D* are supplementary, and therefore $m\angle AOB = 120°$.
5. *GBHO* is a kite, because $\overline{GB}$ and $\overline{GO}$ are radii of circle *G*, and $\overline{HB}$ and $\overline{HO}$ are radii of circle *H*.
6. $m\angle GKO = 90°$, because the diagonals of a kite are perpendicular.
7. *GOIA* is a kite, since $\overline{GO}$ and $\overline{GA}$ are radii of circle *G*, and $\overline{IO}$ and $\overline{IA}$ are radii of circle *I*.
8. $m\angle GJO = 90°$, because the diagonals of a kite are perpendicular.
9. $m\angle KGJ = 360° - 90° - 90° - 120° = 60°$ (using the fact that the angles of quadrilateral *GJOK* sum to 360°).
10. In the same way, it can be shown that one of the other angles is 60° (implying that the remaining one is also 60°).

Further Exploration

Students may discover the following two interesting generalizations:

1. If *similar* triangles *DBA*, *BEC*, and *ACF* are erected on the sides of any triangle *ABC*, their circumcenters *G*, *H*, and *I* form a triangle similar to the three triangles.
2. If *similar* triangles *DBA*, *CBE*, and *CFA* are erected on the sides of any triangle *ABC*, their circumcenters *G*, *H*, and *I* form a triangle similar to the three triangles. (Notice that the similar triangles have different orientations in these two different generalizations.)

You may wish to encourage even further exploration by asking students to consider what happens if the angles *D*, *E*, and *F* are arbitrary, but together sum to 180°, because this latter investigation leads to the following generalization:

3. If triangles *DBA*, *BEC*, and *ACF* are erected on the sides of any triangle *ABC* so that $m\angle D + m\angle E + m\angle F = 180°$, their circumcircles meet in a common point, and their circumcenters *G*, *H*, and *I* form a triangle, then $m\angle G = m\angle D$, $m\angle H = m\angle E$, and $m\angle I = m\angle F$ (see figure on the next page).

Proof

This result can be proved in exactly the same way as for the equilateral triangles earlier. For example, construct circumcircles *ADB* and *BEC* to intersect in *B* and *O* (see figure on the next page). Joining *O* with *A*, *B*, and *C*, we see that $m\angle BOC = 180° - m\angle E$, $m\angle AOB = 180° - m\angle D$, and so on.

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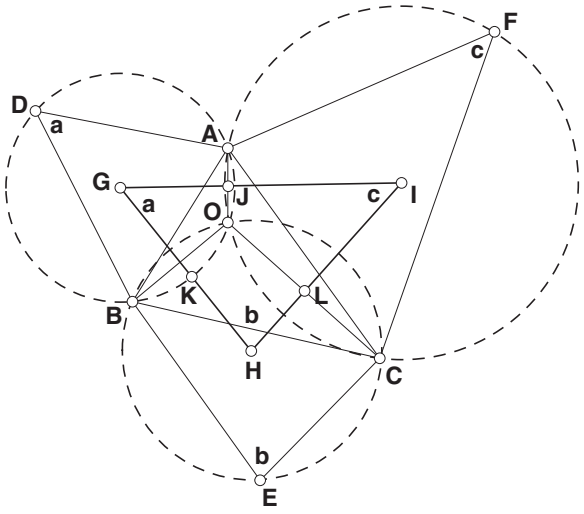


$$m\angle AOC = 360° - (m\angle BOC + m\angle AOB)$$
$$= 360° - (180° - m\angle E + 180° - m\angle D)$$
$$= m\angle E + m\angle D$$
$$= 180° - m\angle F$$

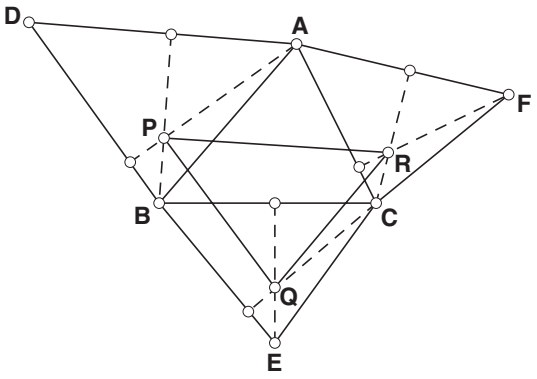
Therefore, $\angle AOC$ and $\angle F$ are supplementary, and O therefore lies on the circumcircle of ΔAFC. Now since $\overline{GH}$, $\overline{HI}$, and $\overline{GI}$ are respectively perpendicular to the common chords *OB, OC,* and *OA,* it follows that *OKGJ, OKHL,* and *OLIJ* are all cyclic quadrilaterals. Therefore, $\angle I$ must be the supplement of $\angle AOC$, but from the aforementioned result, we have $\angle F$ also the supplement of $\angle AOC$. Therefore, $\angle I = \angle F$, and in the same way it follows that $\angle G = \angle D$ and $\angle H = \angle E$.

Note that both generalizations (1) and (2), about similar triangles, are simply special cases of generalization (3).

Another interesting version of the first generalization on the previous page is given in **Napoleon 2.gsp** (proofs can be found in de Villiers 1996, 177–181 and King 1997):

4. If *similar* triangles *DBA, BEC,* and *ACF* are erected on the sides of any triangle *ABC,* and any three points *P, Q,* and *R* are chosen so that they respectively lie in the same positions relative to these triangles, then *P, Q,* and *R* form a triangle similar to the three triangles.

m∠DAB = 54°	m∠QRP = 54°
m∠DBA = 76°	m∠PQR = 76°
m∠BDA = 50°	m∠QPR = 50°



For example, the figure above shows that the respective orthocenters of the similar triangles form another triangle similar to the three exterior triangles. Similarly, the respective centroids and incenters would form a similar triangle.

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